AD-A098 067 MASSACHUSETTS INST OF TECH CAMBRIDGE DEPT OF MATHEMATICS F/G 12/1
THEORETICAL AND NUMERICAL ANALYSIS OF CONFORMAL MAPPING.(U)
JAN 81 M DUBINER AFOSR-77-3405
UNCLASSIFIED AFOSR-TR-81-0392 NL

1 OF 3
AD A
098067

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LEVEL II

(3)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR-81-0392	2. GOVT ACCESSION NO. AD-A098067
4. TITLE (and Subtitle) THEORETICAL AND NUMERICAL ANALYSIS OF CONFORMAL MAPPING	5. TYPE OF REPORT & PERIOD COVERED INTERIM
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Moshe Dubiner	8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3405
9. PERFORMING ORGANIZATION NAME AND ADDRESS M.I.T. Department of Mathematics Cambridge MA 02139	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A3
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB DC 20332	12. REPORT DATE JAN 81
	13. NUMBER OF PAGES 205
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Doctoral thesis	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DTIC ELECTE APR 22 1981 S E

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Many numerical simulations, in particular that of a two dimensional incompressible free boundary flow, can be done by performing conformal mapping of the flow domain onto a half plane. The detailed behavior of the conformal mapping, which is closely related to the detailed behavior of the solution to a two dimensional Dirichlet problem, is analysed. A uniform asymptotic expansion to the conformal map of a slender domain is constructed. Its salient features are explained and later generalized by theorems valid for arbitrary domains. It is

(continued on reverse)

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD A098067

DTIC FILE COPY

...0021

ITEM #20, CONTINUED:

demonstrated that the conformal map onto a disk can not expand distances beyond a certain bound but can be extremely contracting. The logarithm of its derivative is shown to be well behaved. A general perturbation formula from an arbitrary domain to an arbitrary domain which preserves many features of the infinitesmal perturbation formula is derived, and its use is demonstrated on a fractal. These results utilize two estimates, correct up to a constant factor, of the conformal distance and the location of its geodesics.

The above mentioned theory motivates a new numerical method for the direct computation of the conformal map. When the domain's boundary is resolved by N points our method requires O(N) memory locations and O(N) arithmetic operations. Up to a constant factor the memory requirement is the best possible and the operations number is the lowest achieved so far. Both are an O(N) improvement on the only other direct numerical conformal mapping method which can handle complicated domains. Moreover our numerical approximation has the same "exponential decay of influence" as that of the exact problem.

AFOSR-TR- 81-0392

Theoretical and Numerical Analysis of Conformal Mapping

by

Moshe Dubiner

B.Sc., Tel Aviv University (1974)

Accession For
NTIS GRA&I X
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Dist
A

Submitted In Partial Fulfillment
of the Requirements for the
Degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

January 1981

AFOSR-77-3405

Signature of Author ______________________
Department of Mathematics, January , 1981

Certified by ______________________
Steven A. Orszag
Thesis Supervisor

Accepted by ______________________
Chairman, Departmental Committee
on Graduate Students

Approved for public release;
distribution unlimited.

To my parents,

Doba and Mordechai Dubiner

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is
approved for public release IAW AFR 190-12 (7b).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

81 4 22 104

Acknowledgement

I would like to express my deep gratitude to my advisor, Professor Orszag, for suggesting the thesis topic, for his helpful comments and criticism and for his generous assistance in the preparation of this thesis. This work was supported by the Air Force Office of Scientific Research under Grant No. 77-3405 .

Table of Contents

Abstract

Title: Theoretical and Numerical Conformal Mapping

Author: Moshe Dubiner

Submited to the Department of Mathematics on January 1981, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Many numerical simulations, in particular that of a two dimensional incompressible free boundary flow, can be done by performing conformal mapping of the flow domain onto a half plane. The detailed behaviour of the conformal mapping, which is closely related to the detailed behaviour of the solution to a two dimensional Dirichlet problem, is analysed. A uniform asymptotic expansion to the conformal map of a slender domain is constructed. Its salient features are explained and later generalized by theorems valid for <u>arbitrary</u> domains. It is demonstrated that the conformal map onto a disk can not expand distances beyond a certain bound but can be extremely contracting. The logarithm of its derivative is shown to be well behaved. A general perturbation formula from an arbitrary domain to an arbitrary domain which preserves many features of the infinitesmal perturbation formula is derived, and its use is demonstrated on a fractal. These results

utilize two estimates, correct up to a constant factor, of the conformal distance and the location of its geodesics.

The above mentioned theory motivates a new numerical method for the direct computation of the conformal map. When the domain's boundary is resolved by N points our method requires O(N) memory locations and $O(N^2)$ arithmetic operations. Up to a constant factor the memory requirement is the best possible and the operations number is the lowest achieved so far. Both are an O(N) improvement on the only other direct numerical conformal mapping method which can handle complicated domains. Moreover our numerical approximation has the same "exponential decay of influence" as that of the exact problem.

Thesis Supervisor: Steven A. Orszag.

Title: Professor of Applied Mathematics.

0. Introduction.

Many two dimensional physical problems require the solution of Laplace's equation in a complicated domain Ω. One way to solve these problems is to conformally map Ω onto the unit disk $D(0,1)$ or a half plane $D(\infty)$. Once that is done the Poisson kernel provides the solution to the Dirichlet or restricted Neumann boundary value problems. Hilbert's generalization solves a mixture of the two where each applies on part of the boundary (but it doesn't solve the general Neumann boundary condition). Conversely any method of computing the Dirichlet or Neumann solution can be used to calculate the conformal map (see Theorem 5.3) but there is little reason to do it.

There exists a unique conformal mapping f of Ω onto $D(0,1)$ up to specifying $f(v)$ and $\operatorname{Arg} \partial_v f(v)$ for some $v \in \Omega$. Classical complex analysis demonstrates that on the boundary $\partial\Omega$ f is about as smooth as $\partial\Omega$ is and, of course, f is analytic inside. However, $[\partial_u f(u)]^{-1}$ is ill posed in terms of any reasonable norm of Ω even when u is restricted to be well away from $\partial\Omega$. For example take

$$\Omega = \frac{4}{\pi} \operatorname{arctanh}\left[\tanh \frac{\pi l}{4} \cdot D(0,1)\right] \tag{0.1}$$

where the notation means $\Omega = \left\{ \frac{4}{\pi} \operatorname{arctanh}\left[\tanh \frac{\pi l}{4} \cdot z\right] \mid z \in D(0,1) \right\}$.

It is a smooth domain which looks like an ellipse inflated inside a rectangle centered at the origin of length 2ℓ and width $2-\frac{8}{\pi}\operatorname{arctanh}(e^{-\frac{\pi\ell}{2}})$. But the conformal mapping taking Ω to $D(0,1)$ and 0 to 0 is

$$f(u)=\coth\frac{\pi\ell}{4}\cdot\tanh\frac{\pi u}{4} \tag{0.2}$$

so

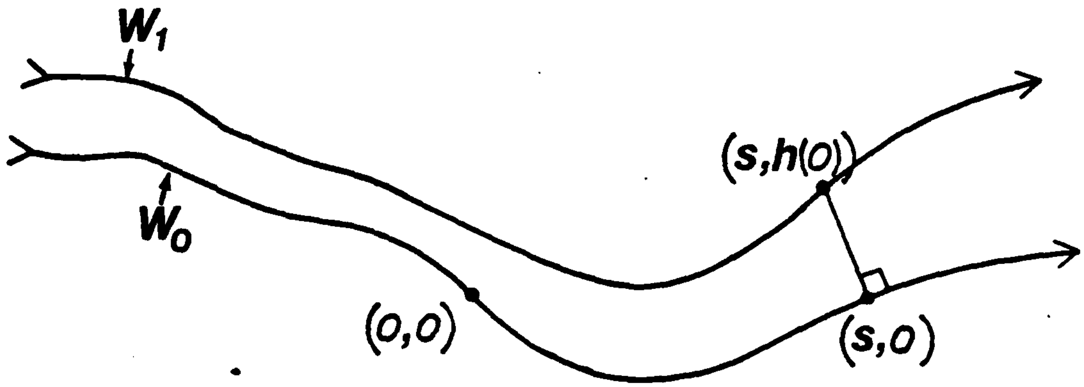
$$\frac{\partial_u f(u)|_{u=\ell}}{\partial_u f(u)|_{u=0}}=\cosh^{-2}\frac{\pi\ell}{4} \tag{0.3}$$

which decreases exponentially in ℓ and equals 0.000000603 for $\ell=10$! The curvature of $\partial\Omega$ near the ends relative to Ω's diameter is $\mathcal{O}(\ell)$ but it is innocent of (0.3). The eccentric cigar shape of Ω is to blame and the same would happen for the smooth paddle-shaped domain of Figure 0.1.

Except near the ends example (0.1) is a slender domain. A domain Ω is called ε slender with ε small (say, $0<\varepsilon<\frac{1}{10}$) iff $\partial\Omega\setminus\{\infty\}$ is composed of two connected components W_0 and W_1 such that for each $u\in W_0$

$$|\kappa(u,W_0)|\cdot|u-\tilde{u}|\leq\varepsilon \tag{0.4}$$

$$\left|\operatorname{Arg}\frac{dW_1|\tilde{u}}{dW_0|u}\right|\leq\varepsilon \tag{0.5}$$



W_1
W_0
$(s,h(0))$
$(0,0)$
$(s,0)$

where $\tilde{u} \in W_1$ minimizes $|\tilde{u} - u|$ and $K(u, W_0)$ is W_0's curvature at u. Condition (0.4) requires W_0 to be nearly straight and condition (0.5) requires W_1 to be nearly parallel to W_0. Parametrize W_0 by its arc length s starting from an arbitrary fixed point. Each $u \in \Omega$ can be uniquely written as

$$u = W_0[s(u)] + t(u)\,\hat{n}[s(u)] \qquad 0 < t < h[s(u)] \tag{0.6}$$

where $\hat{n}(s)$ is the inside unit normal at $W_0(s)$ and $h(s)$ is the distance of $\hat{n}(s)$ from $W_0(s)$ along the direction $\hat{n}(s)$ so that $W_0(s) + h(s)\,\hat{n}(s) \in W_1$. Let us normalize the coordinate system (s, t) in an approximately isotropic way

$$\hat{g}(u) = \frac{\pi}{2}\left[\int_0^{s(u)} \frac{dp}{h(p)} + i\,\frac{t(u)}{h[s(u)]} - \frac{i}{2}\right] \tag{0.7}$$

The map $\hat{g}$ from Ω onto $\Lambda = \{z \mid |\mathrm{Im}\, z| < \frac{\pi}{4}\}$ is quasi-conformal with eccentricity bounded by $c\varepsilon$ (see [7] for definition). Let g be the exact conformal map from Ω onto Λ sending $W_0(\pm\infty)$ to $\pm\infty$ respectively. Clearly

$$f(u) = \tanh g(u) \tag{0.8}$$

conformally maps Ω onto $D(0, 1)$ and so does

$$f(u,v) = \frac{f(u) - f(v)}{1 - f(u)\overline{f(v)}}\, e^{ic(v)} \tag{0.9}$$

and it sends $v \in \Omega$ to 0 . The real number c is determined by

$$\partial_1 f(v,v) > 0 \tag{0.10}$$

where ∂_1 denotes differentiation with respect to the first variable. Formula (0.8) is inserted in (0.9) and results in

$$f(u,v) = \frac{\sinh[g(u) - g(v)]}{\cosh[g(u) - \overline{g(v)}]}\, e^{-i \operatorname{Arg} \partial_v g(v)} \tag{0.11}$$

and (0.11)'s derivative is

$$\partial_u f(u,v) = \frac{\partial_u g(u)}{\cosh^2[g(u) - \overline{g(v)}]} \cos[2 \operatorname{Im} g(v)]\, e^{-i \operatorname{Arg} \partial_v g(v)} \tag{0.12}$$

Define $\tilde{f}(u,v)$ by replacing g with $\tilde{g}$ in (0.11) . It is a quasi-conformal map from Ω onto $D(0,1)$ of at most $c\varepsilon$ eccentricity sending v to 0. Hence $\tilde{f}(\cdot,v)$ is expected to be close to $f(\cdot,v)$ in some sense. Indeed formula (3.3) and others prove that

$$|\ln \partial_u \tilde{f}(u,v) - \ln \partial_u f(u,v)| \leq c\varepsilon\, |\tilde{g}(u) - \tilde{g}(v)| \tag{0.13}$$

where

$$\partial_u = \frac{1}{2}\partial_{Re\,u} - \frac{1}{2} i\,\partial_{Im\,u} \tag{0.14}$$

is defined on nonanalytic functions. Let us press on with the heuristics. Formula (0.12) for $f, \tilde{g}$ shows that

$$\ln|\partial_u f(u,v)| = -\pi \left| \int_{s(v)}^{s(u)} \frac{dp}{h(p)} \right| - \ln h[s(u)] -$$

$$- \ln \sin\left[\pi \frac{t(v)}{h[s(v)]}\right] + O(1) \tag{0.15}$$

$$\operatorname{Arg} \partial_u f(u,v) = -\alpha[0, W_0[s(u)], W_0] + O(1) \tag{0.16}$$

where $\alpha[\,"\,]$ is the change in angle of W_0 between $0 = W_0[s(v)]$ and u's projection on W_0 $W_0[s(u)]$. Thus globally f performs reasonable rotation but extreme scaling. In retrospect it should not be surprising because conformal maps are defined by being locally angle preserving with no scaling restrictions attached.

Formula (0.16) is easy to interpret. It obviously holds (up to translation in $\operatorname{Arg} \partial_u f(u)$ depending on its normalization) for $u \in \partial\Omega$, where Ω is a general domain. Thus (0.16) states that for slender domains

$$\left| \operatorname{Arg} \partial_1 f(\cdot, v)\Big|_u^{\tilde{u}} \right| = O(1) \tag{0.17}$$

where the notation means $|\operatorname{Arg} \partial_{\tilde{u}} f(\tilde{u},v) - \operatorname{Arg} \partial_u f(u,v)| \leq C$ and $\tilde{u} \in \partial\Omega$ is near u, say the closest boundary point. The result (0.17) holds in general as proven by Theorem 5.4. Formula (0.15) is not that easy to generalize. Unlike (0.16), its right side depends on the structure of Ω between v and u. The first question is: what does 'between' mean in general? In order to gain some insight let us consider a more complicated example.

Let $0 < \varepsilon << 1$

$$\Omega(\varepsilon) = \{x+iy \mid \varepsilon x + \cos y > 0\} \tag{0.18}$$

The domain (0.18) has the following property. Any domain Ω is said to be a $\varepsilon > 0$ conjugation of the domains $\{\Lambda_\nu\}_{\nu \in I}$ iff for any $u \in \Omega$ there exists a $\nu \in I$ and two complex numbers a, b such that

$$u \in \Lambda(u) = a\Lambda_\nu + b \tag{0.19}$$

$$\sigma[u, \Lambda(u), \Omega] \leq \varepsilon \tag{0.20}$$

the distance from $\Lambda(u)$ to Ω relative to u is defined by (10.113). The interested reader may prove that any ε slender domain is a $c\varepsilon$ conjugation of

$$\Lambda_1 = \{x+iy \mid |y| < \frac{\pi}{4}\} \tag{0.21}$$

where $c>0$ is constant. Domain (0.18) is a $c\varepsilon$ conjugation of Λ_1 and

$$\Lambda_2 = \{x+iy \mid x - \frac{1}{2}y^2 > 0\} \tag{0.22}$$

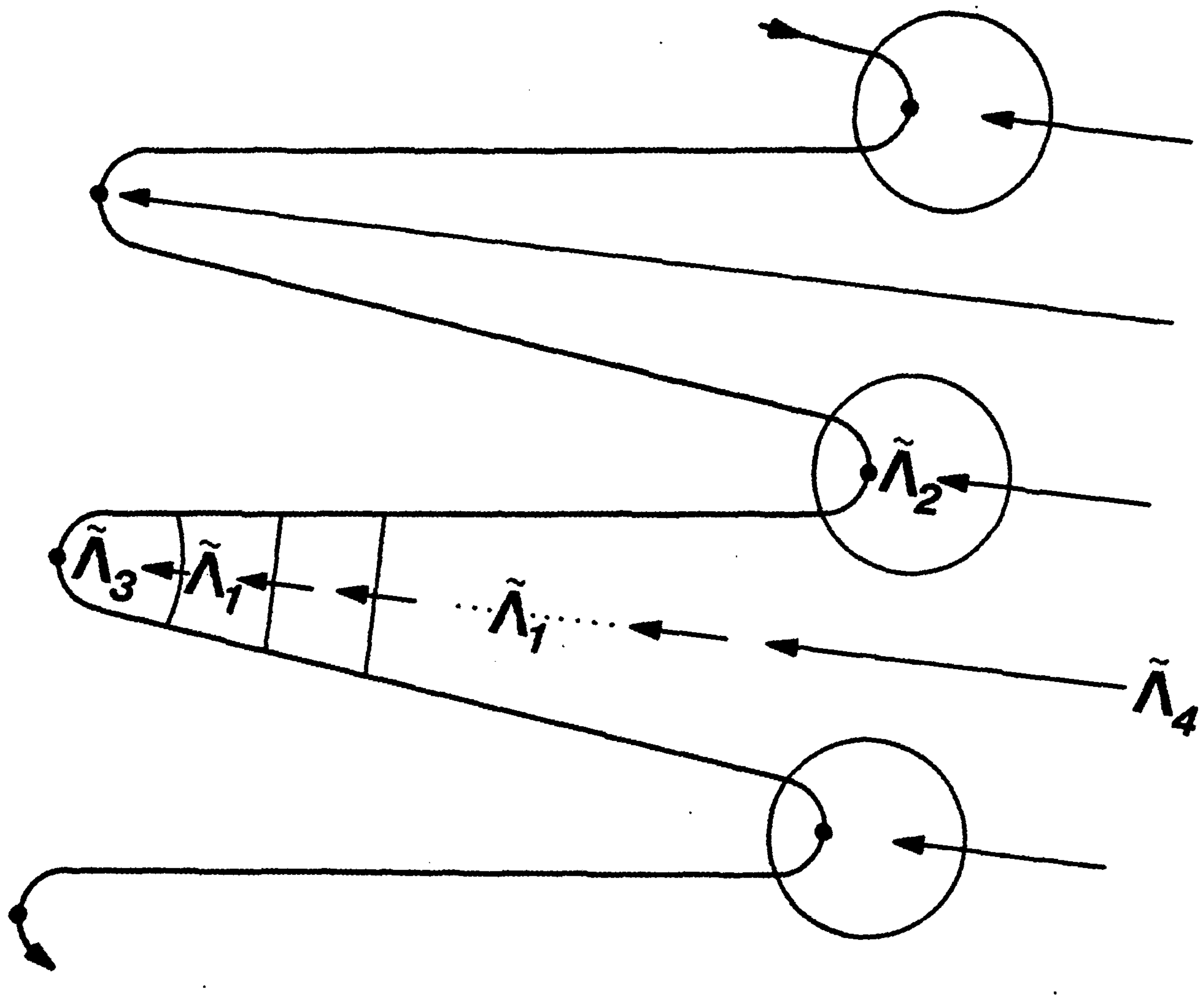
$$\Lambda_3 = \hat{C} \setminus (-\Lambda_2) = \{x+iy \mid x + \frac{1}{2}y^2 > 0\} \tag{0.23}$$

$$\Lambda_4 = \hat{C} \setminus \bigcup_{n=-\infty}^{\infty} [-\infty, i(2n+1)\pi] \tag{0.24}$$

where $[a,b]$ is the closed interval between a and b. We have to match the conformal maps from all the $\Lambda(u)$, $u \in \Omega(\varepsilon)$. In this case it is easiest to do when considering $f[\cdot, \infty, \Omega(\varepsilon)]$, the conformal map from $\Omega(\varepsilon)$ onto the half plane $D(\infty)$ normalized by

$$\partial_1 f(+\infty, \infty) = 1 \tag{0.25}$$

The domain $\Omega(\varepsilon)$ is periodic and symmetric so we can limit ourselves to



$\tilde{\Lambda}_2$
$\tilde{\Lambda}_3$
$\tilde{\Lambda}_1$
$\tilde{\Lambda}_1$
$\tilde{\Lambda}_4$

$$u = x + iy \qquad 0 \le y \le \pi \tag{0.26}$$

We start from

$$f(u, \infty, \Lambda_4) = 2 \operatorname{arcsinh} e^{\frac{1}{2}u} \tag{0.27}$$

It is modified to

$$x > \frac{1}{\varepsilon} - O\left(\frac{1}{\sqrt{\varepsilon}}\right) \qquad \left|x - \frac{1}{\varepsilon} + i(y - \pi)\right| >> \varepsilon \tag{0.28}$$

$$f[x+iy, \infty, \Omega(\varepsilon)] = 2 \operatorname{arcsinh} e^{\frac{1}{2}(x - \frac{1}{\varepsilon}) + iy \frac{\pi}{2h(x)}} \tag{0.29}$$

where

$$h(x) = \begin{cases} \arccos(-\varepsilon x) & -\frac{1}{\varepsilon} \le x \le \frac{1}{\varepsilon} \qquad (0.30) \\ \pi & \frac{1}{\varepsilon} < x \qquad (0.31) \end{cases}$$

and the exact limitation (0.28) will follow from comparison with the following formulas. A priori rigorous bounds can be derived but as usual it is inconvenient. Next

$$f(u, \infty, \Lambda_2) = ic\, e^{2u} \tag{0.32}$$

(there is no natural normalization). We already know how to match the $f[u,\infty,\Lambda(u)]$'s where $\Lambda(u) = a\Lambda_1 + b$: recall (0.7,8). They match with (0.28,29) and give

$$-\frac{1}{\varepsilon} < x < \frac{1}{\varepsilon} + O\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad , \quad \left|x+\frac{1}{\varepsilon}+iy\right| >> \varepsilon \quad , \quad \left|x-\frac{1}{\varepsilon}+i(y-\pi)\right| >> \varepsilon \qquad (0.33)$$

$$f[x+iy,\infty,\Omega(\varepsilon)] = 2e^{\frac{\pi}{2}\left[\frac{Si[h(x)]-Si(\pi)}{\varepsilon} + i\frac{y}{h(x)}\right]} \qquad (0.34)$$

where

$$Si(p) = \int_0^p \frac{\sin q}{q}\,dq \qquad (0.35)$$

Now we can match $f(\cdot,\infty,\Lambda_2)$ to (0.33,34) and obtain

$$\left|x+\frac{1}{\varepsilon}+iy\right| << \varepsilon^{-1/3} \qquad (0.36)$$

$$f[x+iy,\infty,\Omega(\varepsilon)] = 4e^{-\frac{\pi Si(\pi)}{2\varepsilon}} \cos\left[\frac{\pi}{2}\sqrt{1-\frac{2}{\varepsilon}\left[x+\frac{1}{\varepsilon}+iy\frac{\sqrt{2\varepsilon x+2}}{h(x)}\right]}\right] \qquad (0.37)$$

Similarly, $f(\cdot,\infty,\Lambda_3)$ is matched to (0.28,29):

$$\left|x+\frac{1}{\varepsilon}+i(y-\pi)\right| << 1 \qquad (0.38)$$

$$f[x+iy,\infty,\Omega(\varepsilon)] = \sqrt{2\varepsilon}\left(\sqrt{1+\frac{2}{\varepsilon}\left[x-\frac{1}{\varepsilon}+i[y-h(x)]\right]}-1\right) \qquad (0.39)$$

In particular the maximum and minimum of $|\partial_u f[u,\infty,\Omega(\varepsilon)]|$ are obtained at $\frac{1}{\varepsilon}+i\pi,\ -\frac{1}{\varepsilon}$ respectively and

$$\partial_1 f[\tfrac{1}{\varepsilon}+i\pi,\infty,\Omega(\varepsilon)] \sim \sqrt{\frac{2}{\varepsilon}} \qquad (0.40)$$

$$\partial_1 f[-\tfrac{1}{\varepsilon},\infty,\Omega(\varepsilon)] \sim \frac{2\pi}{\varepsilon}\, e^{-\frac{\pi h(\pi)}{2\varepsilon}} \qquad (0.41)$$

What have we learned from example (0.18)? Figure 0.3 illustrates the direction of *information flow* (The reverse of the direction of dependence) which were exhibited while $f[\cdot,\infty,\Omega(\varepsilon)]$ has been constructed. The situration is quite special yet we have some grounds to suspect that in general $\partial_u f(u,v,\Omega)$ and other functions depend mainly on Ω's part 'around' the curve of least Euclidian distance between v and u <u>inside</u> Ω. A close inspection of (0.27-38) reveals that the above mentioned curve from ∞ to u resembles

$$\Gamma(\infty,u,\Omega) = \left\{w\in\Omega \mid 0<\frac{f(w,\infty,\Omega)}{f(u,\infty,\Omega)}<1\right\} \qquad (0.42)$$

The curvces $\Gamma(u,v,\Omega)$ are called geodesics because they are the geodesics of a certain conformally invarient metric

$\rho(u, v, \Omega)$ of Theorem 1.1. Some geodesics of Λ_1 are illustrated in Figure 0.4. Notice for any two $u, v \in \Lambda_1$ far away the most of $\Gamma(u, v, \Lambda_1)$ is exponentially close to Λ_1's axis ot symmetry. Theorem 8.3 demonstrates that in general the geodesics try to keep away from the boundary.

The connection between geodesics and lines of least Euclidian distance is proven in Theorem 9.2.

Now that we have some idea on what 'between' and means it is time to find out how the rest of Ω affects $\ln \partial_u f(u, v, \Omega)$ and other quantities. For that purpose let us return to ε slender domains and compute a next order correction to (0.7). We start by calculating $\partial_{\bar{u}} \tilde{g}$. The gradient ot (0.6) is

$$du = \hat{n}[s(u)]\left[\left(-i + t(u)\, \partial_s \ln \hat{n}[s(u)]\right) ds + dt\right] \tag{0.43}$$

and of (0.7)

$$d\tilde{g}(u) = \frac{\pi}{2h(s)}\left[\left(1 - i t(u)\, \partial_s \ln h(s)\right) ds + i\, dt\right] \tag{0.44}$$

Formulas (0.43,44) combine into

$$d\tilde{g} = \frac{\pi}{4h\hat{n}}\left[2i + \frac{t\partial_s \ln(h\hat{n})}{1+it\partial_s \ln\hat{n}}\right] du -$$

$$- \frac{\pi\hat{n}}{4h}\, \frac{t\partial_s \ln(h\hat{n})}{1+it\partial_s \ln\hat{n}}\, d\bar{u} \tag{0.45}$$

Define the $\tilde{g}$ correction furnction

$$g_1 = g - \tilde{g} \tag{0.46}$$

Then

$$\partial_{\bar{u}} g_1 = -\partial_{\bar{u}} \tilde{g} = -2\pi i \frac{\hat{n}}{h} \mu = -2\pi i\, \eta \tag{0.47}$$

$$\operatorname{Im} g_1 |_{\partial\Omega} = 0 \tag{0.48}$$

where

$$\mu(u) = \frac{1}{8}\, \frac{i t(u)\, \partial_s \ln(h[s(u)]\,\hat{n}[s(u)])}{1+it(u)\, \partial_s \ln \hat{n}[s(u)]} \tag{0.49}$$

Notice that Ω is $\mathcal{O}(\varepsilon)$ slender iff

$$\sup_{u\in\Omega} |\mu(u)| \leq C\varepsilon . \tag{0.50}$$

Problem (0.47,48) has a unique solution up to an additive real constant

$$g_1(u) = -i \iint_\Omega \left[\frac{\eta(w)\,\partial_w f(w)}{f(u)-f(w)} - \frac{\overline{\eta(w)\,\partial_w f(w)}}{\frac{1}{f(u)} - \overline{f(w)}} \right] |d^2 w| \tag{0.51}$$

$$\begin{aligned} g(u) = \tilde{g}(u) - i \iint_\Omega \Big[& \eta(w)\,\partial_w g(w) \coth[g(u)-g(w)] - \\ & - \overline{\eta(w)\,\partial_w g(w)} \tanh[g(u)-\overline{g(w)}] \Big] |d^2 w| \end{aligned} \tag{0.52}$$

where

$$|d^2 w| = d\,\mathrm{Re}\,w \cdot d\,\mathrm{Im}\,w \tag{0.53}$$

Fromula (0.51) is an integral equation of the first kind which can be iterated to convergence. The first order correction to $\tilde{g}$ with some modifications is

$$g(u) \sim U + \iint_\Omega [\mu(w) \coth(U-W) + \overline{\mu(w)} \tanh(U-\overline{W})] |d^2 W| \tag{0.54}$$

where we have and will abbreviate

$$U = \tilde{g}(u), \quad V = \tilde{g}(v), \quad W = \tilde{g}(w) \tag{0.55}$$

Recall (0.11)

$$\ln f(u,v) = \ln \frac{\sinh(U-V)}{\cosh(U-\bar{V})} - \frac{1}{2} \ln \frac{\partial_U V}{\partial_{\bar{U}} \bar{V}} \tag{0.56}$$

so

$$\ln f(u,v) \sim \ln \bar{f}(u,v) +$$

$$+ \iint_{\Omega} \left[\mu(W) K(U,V,W) + \overline{\mu(W)} K\left(U,V,\overline{W} + i\frac{\pi}{2}\right) \right] |d^2 W| \tag{0.57}$$

$$K(U,V,W) = \coth(U-V)\left[\coth(U-W) - \coth(V-W)\right] -$$

$$-\tanh(U-\bar{V})\left[\coth(U-W) - \tanh(\bar{V}-W)\right] +$$

$$+\frac{1}{2}\sinh^{-2}(V-W) - \frac{1}{2}\sinh^{-2}(\bar{V}-W) =$$

$$= \frac{\cos(2\,\mathrm{Im}\,V)}{2\sinh(U-W)\sinh(V-W)\sinh(\bar{V}-W)} \left[\frac{\sinh(U-V)}{\sinh(U-W)} - \frac{\cosh(U-\bar{V})}{\cosh(\bar{V}-W)} \right] \tag{0.58}$$

and

$$\ln \partial_u f(u,v) = \ln f(u,v) + \ln \partial_u \ln f(u,v) \tag{0.59}$$

so

$$\ln \partial_u f(u,v) = \ln \partial_u \tilde{f}(u,v) +$$

$$+ \iint_{\Omega} [\mu(w) L(U,V,W) + \overline{\mu(w)}\, L(U,V,\overline{W}+i\tfrac{\pi}{2})]\, |d^2 w| \tag{0.60}$$

$$L(U,V,W) = K(U,V,W) + \frac{\partial_U K(U,V,W)}{\partial_U \tilde{f}(u,v)} =$$

$$= \frac{1}{2\sinh(U-W)\sinh(V-W)\sinh(\overline{V}-W)} \left[\frac{\cos(2\,\Im V)\sinh(U-V)}{\sinh(V-W)} - \right.$$

$$\left. - \frac{\cos(2\,\Im V)\cosh(U-\overline{V})}{\cosh(\overline{V}-W)} + 2\,\frac{\sinh(U-V)\sinh(U-\overline{V})}{\sinh(U-W)} \right] \tag{0.61}$$

If we assume that $\mu(w)$ is not only of order ε but is also slowly varying, for instance $\frac{1}{h(s)}\left|\partial_s \mu[W_0(s) + h(s)\hat{n}(s)]\right| \le C\varepsilon^2$ then the integration in (0.57,60) can be done explicitly. Of course that results is much easier to derive directly and is of no interest to us. What we have wanted and obtained is the relative dependence of $\partial_u f(u,v)$ on the boundary part $Q = W_0([s-\frac{1}{2}\Delta s, s+\frac{1}{2}\Delta s])$ centered at $w = W_0(s)$ of length Δs. When

$$0 < \Delta s \le h(s) \le |u-w|, |v-w| \tag{0.62}$$

the contrlling factor of Q's influence is

$$e^{-4\inf|\tilde{g}(w) - [\tilde{g}(v), \tilde{g}(u)]|} \sup|\mu(Q)| \frac{\Delta s}{h(s)} \tag{0.63}$$

The asymptotically correct term is the same with $\tilde{g}$ replaced by g. A graphic interpretation of (0.63) follows. In order to affect $\partial_u f(u,v)$ the data about Q's shpae must travel from Q to v. It is provided a free ride in Ω's portion between u and v (in general on the geodesic $\Gamma(u,v)$ between u and v) but it must pay $\frac{\pi}{2h[s(z)]}$ per distance for travel around any point z not between u and v. The data is thrifty so it will move along a geodesic (a cost minimizing curve which turns out to agree with 0.42) to some point σ between v and u and will then enjoy a free ride to v. The optimal choice of σ is the middle point among u, v, w. Let us call the total minimal cost $\rho_x(w, \Delta s, u, v)$. Then when the Q data reaches v its intensity is deminished by a factor of $e^{-4\rho_x}$. A similar situation holds for other function of the conformal map besides $\ln \partial_u f(u,v)$, except that the ride on the geodesic between u and v is not free but on a reduced fare. For instance (0.57,58) shows that the controlling factor of Q's influence on $\ln f(u,v)$ is

$$e^{-2\inf|\tilde{g}(w)-[\tilde{g}(v),\tilde{g}(u)]|-2|\tilde{g}(w)-\tilde{g}(v)|}\sup|\mu(z)|\frac{\Delta s}{\lambda(s)} \quad (0.64)$$

which means that the travel on $\Gamma(u,v)$ is done on a half fare. Notice that this does not change σ, the point of transfer to $\Gamma(u,v)$.

Now that we know what (0.63) means let us understand where it comes from. The term $e^{-2\rho_x}$ is simply the decay of a Dirichlet or Neumann data from Q to $\Gamma(u,v)$ and is already present in (0.54). However, in the computation of (0.58) a cancellation has occurred and another $e^{-2\rho_x}$ has appeared. Lest it look like a freak accident let us derive it from another point of view, close in spirit if not in technical complexity to section 10's. Suppose that Ω is not only ε slender but is ε close to Λ. If w is between u and v than $\rho_x = 0$ so let us consider v between u and w (the remaining case is similar). Define

$$f(z) = \tanh(z - \mathrm{Re}\, v) \qquad (0.65)$$

The domain $f(\Omega)$ is close to the unit disk. The image of Q has length of order $e^{-2\rho_x}\Delta s$. A major change in Q modifies $f(\Omega)$ by a region away from $f(u)$, $f(v)$ whose area is of order $(e^{-2\rho_x}\Delta s)^2$. The details will be presented in Section 10 but it takes no great leap of imagination to conclude that Q's effect is at most proportional to that area, and that is where the extra $e^{-2\rho_x}$ comes from. What about $g(u)$? It is normalized so that $g(\pm\infty) = 1$ and $|f(w)-1|$ or $|f(w)+1|$ is of order $e^{-2\rho_x}$ so Q's influences on $g(u)$ is of order $\frac{(e^{-2\rho_x}\Delta s)^2}{e^{-2\rho_x}}$.

The interpretation of (0.63,64) was chosen so that it is generalizable to arbitrary domains, with some modifications.

The asymptotic theory has to be replaced by estimation up to a constant factor. The measure μ of Ω's deviation from Λ's shape is special and anyway there is no ideal general domain. Instead we will pick a wide challange and consider a perturbation of a general domain Ω to a specified general domain $\tilde{\Omega}$. We will divide the perturbation into parts and prove in Theorem 10.5' that each such part Q of diameter Δs centered at w affects $\ln \partial_n f(u,v)$ by $\delta_\chi^2(w,\Delta s,u,v)$ which is a generalization of (0.63) with $\sup|\mu(Q)|$ replaced by $\frac{\Delta s}{\lambda(s)}$. The term $\delta(w,\Delta s,u,v)$ will be first encountered in Theorem 8.6 and

$$\delta_\chi(w,\Delta s,u,v) = \sup_{\sigma\in\Gamma(u,v)} \delta(w,\Delta s,u,v) \qquad (0.66)$$

The general interaction between u, v, w and σ is described by Theorem 4.6, which should be combined with Theorem 9.2.

The local length scale $\lambda[s(w)]$ will be generalized into $a(w,u,v)$ of (9.2). Its dependence on u,v is unfortunatly unavoidable: consider a half plane $\Omega = D(\infty)$. The lack of a local length scale, except of $\inf|w-\partial\Omega|$ which vanishes on the boundary is the most important general indegredient missing in slender domains. It is not fortuitous that the conformal metric $\rho(u,v)$ blows up at the boundary.

Our examples were mostly of smooth domains but notice that domain (0.18) has a very sharp bend at $\frac{1}{\varepsilon}+i\pi$, which coerresponds to Λ_4's corner, and it did not distrub us from completing a uniform asymptotic approximation. Theorem (10.7) proves an interesting property of a fractal, which is the applied mathematician's ultimate in roughness. However, throghout this thesis we insist on obtaining specific estimates at specific points, in contrast to the `average' type approach of P.D.E. Theory. That is an advantage when it works but it fails near a rough boundary. In Section 5 it is shown how to patch our results with P.D.E. Theory. Some synthesis is clearly required.

Let us now consider the numerical computation of conformal mappings. As remarked earlier, any Laplace solver will do. Suppose that the domain Ω is covered by $\mathcal{O}(N^2)$ points, N of which are on the boundary $\partial\Omega$. Then a Laplace solver requires storing $\mathcal{O}(N^2)$ numbers and performing from $\mathcal{O}(N^2 \ln N)$ to $\mathcal{O}(N^3)$ operations, where the last estimate is more realistic for complicated domains. The grid set up is troublesome, especialy for multiscaled and time dependent (free boundary) domains.

One way to avoide an internal grid is by using a vortex represontation. That results in an integral equation of the first kind which is numerically formulated as a set of $N \times N$ linear equations. It can be solved by by Gaussian elimination which requires $\mathcal{O}(N^2)$ memory locations and $\mathcal{O}(N^3)$ operatios. Alternativly one may iterate the system using $\mathcal{O}(N)$ memory and

$\mathcal{O}(N^2)$ operations per iteration. This is the numericl approximation to Neumann's series, and the later is gaurenteed to converge for any single sheeted domain satisfying some mild conditions. The best existing Rayleigh-taylor instability simulation has been done in that way by Baker, Meiron and Orszag [12]. It is relatively easy to program and generelizes to 3 dimensions. Moreover it can handle two incompressible fluids problems which conformal mapping alone can not solve unless the fluids density ratio is 0 or 1. However Neumann's series's convergence is precarious. The rate of convergence for the domain Ω equals that for its exterior domain $\hat{C}\setminus\Omega^c$ and in particular convergence fails for multisheeted domains and is very slow when two separate parts of $\partial\Omega$ approach each other. The domain (0.18) requires $\mathcal{O}(\frac{1}{\varepsilon})$ iterations per order reduction in the error. Moreover the Neumann series seems hard to modify in a way which will extract a singularity such as a corner and still preserve convergence for general domains.

The most natural numerical conformal mapping computation is done by taylor expanding the conformal function from the unit disk onto Ω. Several such methods are listed in [9]. The best of them takes only $\mathcal{O}(N)$ memory locations and $\mathcal{O}(N^2 \ln N)$ operations but we have seen that the series will not converge to domain (0.18) before $N = e^{9/\varepsilon}$ terms are taken.

The first dirct computation of the conformal map onto a cigar shaped domain has been done by Manikoff and Zemack [ ]. Their method is to set up a system of $N \times N$ nonlinear equatins and solve them by Newton iteration. Each iteration takes $\mathcal{O}(N^2)$ memory and $\mathcal{O}(N^3)$ operations and only few iterations are required.

Any partial differential equation on a time dependent domain can be solved by a Green's function method which utilizes only boundary data. This is not usually done because it takes $\mathcal{O}(N^{2d-2})$ memory locations and $\mathcal{O}(N^{2d-2})$ operations in d dimensions which is unresonable for $d \geq 3$. For the ∇^2 operator in 2 dimensions better can be done because then the Green's function $G(u, v, \Omega)$ is consructable from $G(u, v_0, \Omega)$ and its harmonic conjugate where v_0 is constant. That is the basis of our method, though it will be presented in a different way. It takes $\mathcal{O}(N)$ memory locations and $\mathcal{O}(N^2)$ operations. Its other virtues will be described in section 11.

1. The Conformal Metric.

Throughout this thesis Ω is assumed to be a simply connected open subset of the compact (with ∞) complex plane such that $\hat{\mathbb{C}}\setminus\Omega$ contains more than one point, unless stated otherwise. The Riemann mapping theorem states that Ω can be conformally mapped onto any other domain satisfying the above mentioned requirements. For each $\vartheta\in\Omega$ let $f(\cdot,\vartheta,\Omega)$ conformally map Ω onto the unit disc and send ϑ to the center:

$$f(\Omega,\vartheta,\Omega) = D(0,1) = \{z \mid |z| < 1\} \tag{1.1}$$

$$f(\vartheta,\vartheta,\Omega) = 0 \tag{1.2}$$

The mapping function f is unique up to rotation. The necessity of specifying ϑ is a nuisance but is also a very useful theoretic1 tool because it allows us to focus on any part of Ω at will.

The following theorem is the cornerstone of our approach. It is original in spirit though the results are classical except for (1.3).

<u>Theorem 1.1</u>: For any $u, v \in \Omega$

$$|\partial_u f(u,v,\Omega)| = \frac{F(u,\Omega)}{\cosh^2 \rho(u,v,\Omega)} \tag{1.3}$$

$$|f(u,v,\Omega)| = \tanh \rho(u,v,\Omega) \tag{1.4}$$

where

$$F(u,\Omega) = |\partial_1 f(u,v,\Omega)| \tag{1.5}$$

$$\rho(u,v,\Omega) = \min_{\Gamma \in S^c(u,v,\Omega)} \int_\Gamma F(z,\Omega)\,|dz| \tag{1.6}$$

$$S^c(u,v,\Omega) = \{\text{curve } \Gamma \mid \Gamma \subset \Omega^c \quad u,v \in \Gamma^c\} \tag{1.7}$$

The function $\rho(u,v)$ is a conformally invariant metric. In particular, it is symmetric and satisfies the triangle inequality. The functions $\ln F(u)$ and $F(u)$ are subharmonic and satisfy

$$\nabla^2 \ln F = 4F^2 \tag{1.8}$$

The open geodesic $\Gamma(u,v)$ exists, is unique, analytic and is characterized by

$$\partial_n \ln F = -\kappa \qquad \text{on } \Gamma \tag{1.9}$$

where n is the normal and κ is its curvature of Γ. The global geometry under distance ρ is Lobachevski's hyperbolic geometry. Thus, for any $u, v, w \in \Omega$ $f[\Gamma(u,v), w]$ is a circular arc orthogonal to the unit circle.

<u>Proof</u>: Clearly

$$f(u,v,\Omega) = f[f(u,w,\Omega), f(v,w,\Omega), D(0,1)] \tag{1.10}$$

$$f[p,q,D(0,1)] = \frac{p-q}{1-p\bar{q}} \tag{1.11}$$

Differentiation with respect to u gives

$$\partial_u f(u,v,\Omega) = \partial_u f(u,w,\Omega) \frac{1-|f(v,w,\Omega)|^2}{[1-f(u,w,\Omega)\overline{f(v,w,\Omega)}]^2} \tag{1.12}$$

$$F(u,\Omega) = |\partial_u f(u,v,\Omega)| \, [1-|f(u,w,\Omega)|^2]^{-1} \tag{1.13}$$

Obviously

$$\frac{d}{|dz|}|f(z)| \leq |\partial_z f(z)| \tag{1.14}$$

with equality iff

$$\operatorname{Arg} \partial_z f(z) = \operatorname{Arg} f(z) \tag{1.15}$$

Thus

$$\frac{d|f(z)|}{|dz|}[1-|f(z)|^2]^{-1} \leq F(z) \tag{1.16}$$

$$\frac{d}{|dz|} \operatorname{arctanh} |f(z)| \leq F(z) \tag{1.17}$$

Integrating on any $\Gamma \in S^c(u,v)$ provides

$$\operatorname{arctanh} |f(z)| \leq \int_\Gamma F(z)\,|dz| \tag{1.18}$$

Equality holds iff (1.15) is satisfied on all Γ, which happens iff $f(\Gamma)$ is the straight line between 0 and $f(u)$. We have just proven (1.4) and characterized the geodesics which pass through v. The generl geodesics are obtained by (1.10,11). Inserting (1.4) in (1.13) gives (1.3).

Formula (1.9) is the two dimensional case of Euler's equation of geometrical optics. In order to obtain (1.8) take the logarithm of (1.13).

$$\ln F = \mathrm{Re}\, \ln \partial_1 f - \ln(1 - e^{2h}) \tag{1.19}$$

where

$$h = \mathrm{Re}\, \ln f \tag{1.20}$$

Hence

$$\nabla h = (\mathrm{Re}\, \partial_1 \ln f,\ \mathrm{Im}\, \partial_1 \ln f) \tag{1.21}$$

$$|\nabla h| = \left|\frac{\partial_1 f}{f}\right| = (e^{h} - e^{-h}) F \tag{1.22}$$

and

$$\nabla \ln(1 - e^{-2h}) = \frac{2}{1 - e^{-2h}} \nabla h \tag{1.23}$$

$$\nabla^2 \ln F = \frac{4 e^{-2h}}{(1 - e^{-2h})^2} |\nabla h|^2 = 4 F^2 \tag{1.24}$$

While proving (1.18) we haven't actually relied on the univalence of f or on its being <u>onto</u> $D(0,1)$. Thus we have proven Schwarz's lemma which is formula (1.28) of the following amalgamation of monotonicity results about F, ρ and the harmonic measure

$$\omega(W, v, \Omega) = \text{Length } f(W, v, \Omega) \qquad W \in \partial\Omega \tag{1.25}$$

<u>Theorem</u> <u>1.2</u>: Suppose that g is a nonconstant analytic function from Ω_1 into Ω_2

(1.26)

and $v \in \Omega_1$. Then

$$F(v, \Omega_1) \geq |\partial_v g(v)| \; F[g(v), \Omega_2] \tag{1.27}$$

and equivalently for any $u, v \in \Omega_1$

$$\rho(u, v, \Omega_1) \geq \rho[g(u), g(v), \Omega_2] \tag{1.28}$$

For any $w \in g(\partial\Omega_1) \cap \partial\Omega_2$

$$\sum_{z \in \text{inv}\, g(\{w\})} \left| \frac{d f(z, v, \Omega_1)}{d g(z)} \right| \leq \left| \frac{d f[w, g(v), \Omega_2]}{dw} \right| \tag{1.29}$$

and equivalently for any $W \in g(\partial\Omega_1) \cap \partial\Omega_2$

$$\omega[\operatorname{inv} g(W), v, \Omega_1] \leq \omega[W, g(v), \Omega_2] \tag{1.30}$$

<u>Proof</u>: Formula (1.27) is well known and trivial. It combines with

$$g[S(u, v, \Omega_1)] \subset S[g(u), g(v), \Omega_2] \tag{1.31}$$

to reprove (1.28). In turn (1.28) can be inserted in (1.3) and gives (1.29) with the left hand sum restricted to a single z which is satisfactory when g is univalent. Formula (1.30) is the integral of (1.29).

Here follows the general proof of (1.29). Normalize

$$\Omega_1 = \Omega_2 = D(\infty) = \{z \mid \operatorname{Re} z > 0\} \tag{1.32}$$

$$v = g(v) = 1 \quad , \quad w = 0 \tag{1.33}$$

Because $g[D(\infty)] \subset D(\infty)$ it follows that for any $z \in D(\infty)$ such that $g(z) \in \partial D(\infty)$

$$z \in \partial D(\infty) \tag{1.34}$$

$$\partial_z g(z) \geq 0 \tag{1.35}$$

Moreover for any $z \in \partial D(\infty)$

$$|\partial_z f[z, 1, D(\infty)]| = \frac{2}{|1+z|^2} = \frac{2}{1-z^2} \tag{1.36}$$

Thus (1.29) is rewritten as

$$\sum_{z \in \operatorname{inv} g(\{\omega\})} \frac{dz}{dg(z)} \frac{1}{1-z^2} \leq 1 \tag{1.37}$$

Clearly

$$1 - \sum{}'' = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} \frac{1}{g(z)} \frac{dz}{1-z^2} \tag{1.38}$$

and for $z \in \partial D(\infty)$

$$\operatorname{Re} g(z) \geq 0 \quad , \quad \frac{dz}{i(1-z^2)} > 0 \tag{1.39}$$

so

$$\operatorname{Re}\left[1 - \sum{}''\right] \geq 0 \tag{1.40}$$

which proves (1.37).

In most applications of theorem 1.2 is the identity map or the cover map of a multisheeted domain, which will be defined in section 3.

2. Conformal Mapping onto a Half Plane.

We want to extend $f(\cdot, v, \Omega)$ to v's in the boundary. But $\partial\Omega$ is not always the right set to study. For example the point $v = 1$ on the boundary of $\hat{C} \setminus \{z \mid z > 0\}$ is really two points: $0 + 0 \cdot i$ and $0 - 0 \cdot i$. The simplest solution to our problem is to let some $f(\cdot, \omega, \Omega)$ distinguish between the points. The set $\Omega^c = \Omega \cup \partial\Omega$ is the smallest Ω completion which has a continuous extension of the identity map $id : \Omega \to \hat{C}$. Similarly let $\Omega^b = \Omega \cup \tilde{\partial}\Omega$ be the smallest Ω completion with continuous extensions of both id and $f(\cdot, \omega, \Omega)$ for some, and hence all, $\omega \in \Omega$. The set $\tilde{\partial}\Omega$ can be represented as

$$\tilde{\partial}\Omega = \{ \tilde{v} \mid \tilde{v} = (v, f(v)),\ v \in \Omega\ ;$$

$$;\ \exists \{v_j\} \subset \{\Omega\},\ v_j \to v,\ f(v_j) \to f(\tilde{v}) \} \qquad (2.1)$$

For each $v \in \tilde{\partial}\Omega$, let $f(\cdot, v, \Omega)$ be the conformal map of Ω onto a half plane whose continuous extension to Ω^b sends v to ∞:

$$f(\Omega, v, \Omega) = D(\infty) = \{z \mid Re\, z > 0\} \qquad (2.2)$$

$$f(v, v, \Omega) = \infty \qquad (2.3)$$

This determines f up to scaling and translation. The maps $f(\cdot, \upsilon, \Omega)$ where $\upsilon \in \Omega$ match those where $\upsilon \in \partial\Omega$:

<u>Theorem 2.1</u>: For any $\tilde{\upsilon} \in \partial\Omega$, there exist two real functions β and γ such that for each $u \in \Omega$, $\Omega \ni \upsilon \to \tilde{\upsilon}$

$$f(u,\upsilon) = e^{i\gamma(\upsilon)}\left[1 - \beta(\upsilon) f(u,\tilde{\upsilon}) + o(\beta(\upsilon))\right] \qquad (2.4)$$

$$0 < \beta(\upsilon) \to 0 \qquad (2.5)$$

<u>Proof</u>: Take some $w \in \Omega$. Rearranging (1.10.11) results in

$$f(u,\upsilon) = -e^{i\alpha} + (1-r)\frac{f(u,w) + e^{i\alpha}}{1 - f(u,w)\, r\, e^{-i\alpha}} \qquad (2.6)$$

where

$$r e^{i\alpha} = f(\upsilon, w) \quad , \quad r > 0 \qquad (2.7)$$

Let $\upsilon \to \tilde{\upsilon}$. Then $f(\upsilon,w) \to f(\tilde{\upsilon}, w)$ so $r \to 1, \alpha \to \alpha_0$ so

$$\frac{1 + e^{-i\alpha} f(u,w)}{1 - r e^{-i\alpha} f(u,w)} \to K[f(u,w)] = \frac{1 + e^{-i\alpha_0} f(u,w)}{1 - e^{-i\alpha_0} f(u,w)} \qquad (2.8)$$

The function K conformaly maps $D(0,1)$ onto $D(\infty)$ and

$f(\vartheta, w) = e^{i\alpha_0}$ to ∞ so

$$K[f(u,w)] = a f(u,\vartheta) + ib \tag{2.9}$$

where a is a scaling constant and b is a translation constant. Combining (2.8,9) with (2.6) proves (2.4) with

$$\beta = a(1-\lambda) \tag{2.10}$$

$$\gamma = \pi + \alpha_0 - b(1-\lambda) + \text{rotation factor} \tag{2.11}$$

The rotation factor in (2.11) is associated with f because f is not unique.

Theorem 1.1 is easily extended to the boundary.

<u>Theorem</u> <u>2.2</u>: For any $u, w \in \Omega$ $\quad v \in \Omega^b$

$$\left|\frac{\partial_w f(w,v)}{\partial_u f(u,v)}\right| =$$

$$= \frac{F(w)}{F(u)} e^{-2\phi(u,w,v)} \left[\frac{1 + e^{-2\rho(u,v)}}{1 + e^{-2\rho(w,v)}}\right]^2 \tag{2.12}$$

$$\phi(u,w,v) = \rho(w,v) - \rho(u,v) \qquad v \in \Omega \tag{2.13}$$

$$\phi(u,w,\tilde{v}) = \lim_{\Omega \ni v \to \tilde{v}} \phi(u,w,v) \qquad \tilde{v} \in \partial\Omega \tag{2.14}$$

and the limit exists alttough $\rho(u,\tilde{v}) = \infty$. For any $u_0 \in \Gamma(u,v)$, $w_0 \in \Gamma(w,v)$

$$\phi(u,w,w_0) \leq \phi(u,w,v) \leq \phi(u,w,u_0) \tag{2.15}$$

in particular

$$|\phi(u,w,v)| \leq \rho(u,w) \tag{2.16}$$

Suppose $\tilde{v} \in \partial\Omega$. Then

$$\frac{\operatorname{Re} f(w,\tilde{v})}{\operatorname{Re} f(u,\tilde{v})} = e^{2\phi(u,w,\tilde{v})} \tag{2.17}$$

<u>Proof</u>: When $v \in \Omega$ formula (2.12) is an immediate consequence of Theorem 1. Suppose $\tilde{v} \in \partial\Omega$ and let $\Omega \ni v \to \tilde{v}$. Formula 1.4 implies that

$$\rho(u,v) = \frac{1}{2} \ln \frac{1+|f(u,v)|}{1-|f(u,v)|} \tag{2.18}$$

according to Theorem 2

$$|f(u,v)| = 1 - \beta(v)\, \mathrm{Re}\, f(u,\tilde{v}) + \mathcal{O}(\beta(v)) \tag{2.19}$$

so

$$\rho(u,v) = \frac{1}{2} \ln \frac{2}{\beta(v)\, \mathrm{Re}\, f(u,\tilde{v})} + \mathcal{O}(1) \tag{2.20}$$

Thus

$$\phi(u,w,v) = \frac{1}{2} \ln \frac{\mathrm{Re}\, f(u,\tilde{v})}{\mathrm{Re}\, f(w,\tilde{v})} + \mathcal{O}(1) \tag{2.21}$$

which proves the existence of $\phi(u,w,\tilde{v})$ and formula (2.17). Formulas (2.12,14) for $v \in \Omega$ imply (2.12) for $v \in \tilde{\partial}\Omega$. Formula (2.15) follows from the triangle inequality for the metric ρ.

We had to consider ratios of f at different points beause in general there is no natural normalization of $f(\cdot,\tilde{v})$. When $\infty \in \tilde{\partial}\Omega$ and the boundary is smooth near ∞ so that the following derivative exists, one normalizes by

$$|\partial_{(1)} f(\infty,\infty,\Omega)| = 1 \tag{2.22}$$

3. Estimates of F.

In order to apply theorem 1 we need to estimate F as in

$$F_1(\zeta) \leq F(\zeta) \leq F_2(\zeta) \tag{3.1}$$

Once suitable estimates (3.1) are obtained, we get

$$\inf_{P_1 \in S(u,v)} \int_{P_1} F_1(\zeta) |d\zeta| \leq \rho(u,v) \leq$$

$$\leq \int_{P_2} F_2(\zeta)\, d\zeta \qquad \forall\ P_2 \in S(u,v) \tag{3.2}$$

When Ω is symmetric with respect to reflection in the straight line betweenn u and v the minimization over curves is unnecessary because then $\Gamma(u,v)$ is that line and it does not matter whether it minimizes $\int F_1$ or not. For slender domains one can construct a $\varepsilon \ll 1$ eccentric quasi conformal $\zeta(\cdot, v, \Omega)$ onto $\Lambda = \{\zeta \mid |\mathrm{Im}\, \zeta| < \frac{\pi}{4}\}$. Then the minimization can be avoided:

$$\frac{1-\varepsilon}{1+\varepsilon} \inf_{\zeta \in \Omega} \left[\frac{F_1[\zeta, \Omega]}{F[\zeta(\zeta), \Lambda] \cdot |\partial_\zeta \zeta(\zeta)|} \right] \rho[\zeta(u), \zeta(v), \Lambda] \leq$$

$$\leq \rho(u,v,\Omega) \quad . \tag{3.3}$$

and a similar upper bound holds.

There exist several classical results concerning $F(z)$. Some of these results will be stated as theorems 3.1,4. The connection to the general problem has not been used previously to the best of my knowledge.

Theorem 3.1: For any $v \in \Omega$ where Ω is allowed to be multisheeted with at most n sheets

$$\frac{1}{4n} \sup |\partial\Lambda - \partial\Lambda| \leq F(v,\Omega) \leq$$

$$\leq \inf\{a>0 \mid \exists\, o \in C \quad \Lambda \subset D(o,a)\} \leq \frac{1}{\sqrt{3}} \sup |\partial\Lambda - \partial\Lambda| \quad (3.4)$$

$$\Lambda = \mathrm{Inv}(v,\Omega) = \frac{1}{\hat{C} \setminus (\Omega - v)} \quad (3.5)$$

Conjecture:

$$F(v,\Omega) \leq \frac{2^{7/3}}{9} \sup |\partial\Lambda - \partial\Lambda| \quad (3.6)$$

Equality holds iff Λ is the regular circular 3-gon of angle $\frac{2\pi}{3}$.

$$\frac{2^{7/3}}{9} \doteq 0.560 < \frac{1}{\sqrt{3}} \doteq 0.577 \quad (3.7)$$

Corollary 3.2: Suppose that $\vartheta \in \Omega \neq \infty$. Then

$$\frac{1}{4\, r(\vartheta,\Omega)} \leq F(\vartheta,\Omega) \leq \frac{1}{r(\vartheta,\Omega)} \tag{3.8}$$

$$r(\vartheta,\Omega) = \inf |\partial\Omega - \vartheta| \tag{3.9}$$

The left side inequality of (3.8) is known as the 1/4 circle theorem. Thus even such a crude geometrical consideration provides F up to a factor of 2. The advantage of $Inv(\vartheta,\Omega)$ over the pair (ϑ,Ω) for the purpose of estimating $F(\vartheta,\Omega)$ is apparent.

A slight sharpening of corrolary 3.2 will be useful latar.

Theorem 3.3: For any $\vartheta \in \Omega \neq \infty$

$$F(\vartheta,\Omega) \geq \frac{1}{4\, r(\vartheta,\Omega)} \left[1 + \frac{r(\vartheta,\Omega)}{\sup|\partial\Omega - \vartheta|} \right]^2 \tag{3.10}$$

Proof: Normalize $\vartheta = 0$ and

$$r(0,\Omega) = u > 0 \qquad u \in \partial\Omega \tag{3.11}$$

Denote

$$a = \sup|\partial\Omega| \tag{3.12}$$

$$g(z) = Z[(1)\,\mathrm{inv}\, f(z,0,\Omega)] \tag{3.13}$$

$$Z(p) = \frac{p}{(1+\frac{p}{a})^2} \tag{3.14}$$

The function Z is univalent from $D(0,a)$ thus g is univalent from $D(0,1)$ onto $Z(\Omega)$. By Corollary 3.2

$$F^{-1}[0, Z(\Omega)] \le 4R[0, Z(\Omega)] \tag{3.15}$$

$$F^{-1}(0,\Omega) \le \frac{4u}{(1+\frac{u}{a})^2} \tag{3.16}$$

Accurate F bounds are obtainable by applying theorem 1.2 to $\Omega_1 \subset \Omega_2 \subset \Omega_3$ where Ω_1, Ω_2 are known. Requiring Ω_2 to be simply connected may be very inconvenient because it must depend on Ω's global structure. Instead we can take a multiconnected Ω_2 and define

$$F_+(\vartheta, \Omega_2) =$$

$$= \inf\{F(\vartheta, \tilde{\Omega}) \mid \tilde{\Omega} \subset \Omega_2,\ \tilde{\Omega} \text{ simply connected}\} \tag{3.17}$$

Corollary 3.2 is of that type with

$$\Omega_2 = \hat{\mathbb{C}} \setminus \{\infty, p\} \qquad p \in \partial\Omega \wedge D[\vartheta, \lambda(\vartheta, \Omega)] \tag{3.18}$$

The general stationary condition on $\tilde{\Omega}$ is that it equals Ω_2 minus some curves connecting $\hat{\mathbb{C}} \setminus \Omega_2$'s components and $|\partial_1 f(\cdot, \vartheta, \tilde{\Omega})|$ is continuous across these curves. It is a hard problem even for simple domains.

An easier approach is to cover Ω_2 with a simply connected multisheeted domain

$$\Omega_{2*} = \zeta(\vartheta, \Omega_2, \Omega_2)/R \tag{3.19}$$

where $\vartheta \in \Omega_2$ is arbitrary and R is the homotopy relation between curves in Ω_2 with fixed endpoints. The cover map

$$T_{cov}(\cdot, \Omega_2) : \Omega_{2*} \to \Omega_2 \tag{3.20}$$

simply sends each curve to its endpoint. For example an annulus is covered by a helix and in accordance with Theorem

1.2

$$F[z,(D(0,1)\setminus\{0\})_*] = \frac{1}{2|z|\ln\frac{1}{|z|}} >$$

$$> \frac{1}{1-|z|^2} = F[z, D(0,1)] \tag{3.21}$$

Lower bounds and a useful frame of reference are provided by the capacity inequality.

<u>Theorem</u> <u>3.4</u>: For any $\vartheta \in \Omega$

$$F(\vartheta,\Omega) = e^{-V[Inv(\vartheta,\Omega)]} \tag{3.22}$$

$$V(\Lambda) = \inf_{\eta\in T(\Lambda)} \iint_{\Lambda^2} \ln \frac{1}{|w_1-w_2|}\, d\eta(w_1)\, d\eta(w_2) \tag{3.23}$$

$$T(\Lambda) = \{\text{measure on } \Lambda\ \ \eta \mid d\eta \geq 0\ ,\ \eta(\Lambda) = 1\} \tag{3.24}$$

When $\partial\Omega$ is a Jordan curve a minimal η exists, is supported on $\partial\Lambda$ and equals there

$$\eta(W) = \frac{1}{2\pi}\omega\left(\frac{1}{W}+\vartheta, \vartheta, \Omega\right) \tag{3.25}$$

Formulas (3.22,23) have a physical interpretation. Charge lines of total charge 1, perpendicular to the complex plane are distributed in $\mathrm{Im}\,u(v,\Omega)$ according to η . They arrange themselves so as to minimize the total energy V . The resulting potential at $u\in\hat{C}$ is

$$g(\infty,v,\Omega) = \begin{cases} 0 & u\in\Lambda \quad (3.26) \\ G(\frac{1}{u}+v,v,\Omega) & u\in\hat{C}\setminus\Lambda \quad (3.27) \end{cases}$$

where the Green's function is

$$G(z,v,\Omega) = \ln|z-v| + g(z,v,\Omega) \tag{3.28}$$

$$G(\cdot,v,\Omega)|_{\partial\Omega} = 0 \tag{3.29}$$

and $g(z,v,\Omega)$ is harmonic in $z,v\in\Omega$. Notice that

$$|f(z,v,\Omega)| = e^{G(z,v,\Omega)} \tag{3.30}$$

$$F(v,\Omega) = e^{g(v,v,\Omega)} \tag{3.31}$$

Theorem 3.1 follows easily and "naturally" from theorem 3.4.

Let us prove a distortion Theorem:

<u>Theorem</u> <u>3.5</u>: For any $\vartheta \in \Omega \neq \infty$, $0 \le \alpha \le 1$

$$D[0, \frac{\alpha}{(\sqrt{1+\alpha}+1)^2}] \subset f[D(\vartheta, a), \vartheta] \subset D[0, \alpha] \tag{3.32}$$

$$D[0, \frac{a F(\vartheta)}{1+\gamma}] \subset f[D(\vartheta, a), \vartheta] \subset D[0, a F(\vartheta)(1+\gamma)] \tag{3.33}$$

$$\sup_{u \in D(\vartheta, a)} | \operatorname{Arg} f(u, \vartheta) - \operatorname{Arg}[(u-\vartheta)\, \partial_1 f(\vartheta, \vartheta)] | \le$$

$$\le \frac{4 \ln 4}{\pi} \operatorname{arctanh} \alpha \tag{3.34}$$

where

$$a = \alpha\, \rho(\vartheta) \tag{3.35}$$

$$\gamma = 4^{\frac{4}{\pi} \arctan \alpha} - 1 \tag{3.36}$$

<u>Proof</u>: Normalize $\vartheta = 0$, $\rho(\vartheta) = 1$. By definition $D(0,1) \subset \Omega$ so for any $u \in \partial D(0, \alpha)$

$$\rho(u,0,\Omega) \leq \rho[u,0,D(0,1)] \tag{3.37}$$

$$\operatorname{arctanh} |f(u,0,\Omega)| \leq \operatorname{arctanh} \alpha \tag{3.38}$$

which proves the right hand inequality of (3.32). The left hand inequality is obtained as follows:

$$F^{-1}(u,\Omega) \leq 4\Lambda(u,\Omega) \leq 4(1+|u|) \tag{3.39}$$

$$\rho(u,0,\Omega) > \frac{1}{4}\int_0^\infty \frac{dx}{1+x} = \frac{1}{4}\ln(1+\alpha) \tag{3.40}$$

In order to prove (3.33,34) define the analytic function

$$g(u) = \ln \frac{f(u,0,\Omega)}{k F(0,\Omega)} \tag{3.41}$$

For any $u \in \partial D(0,1)$

$$\operatorname{Re} g(u) \leq -\ln F(0,\Omega) \leq \ln 4 \tag{3.42}$$

$$F^{-1}(0, f[D(0,1),0,\Omega]) \leq 4\Lambda(0, f[\,''\,]) \tag{3.43}$$

$$F(0,\Omega) \leq 4|f(u,0,\Omega)| \tag{3.44}$$

$$\mathrm{Re}\, g(u) \geq -\ln 4 \tag{3.45}$$

In conclusion

$$g[D(0,1)] \subset \Lambda = \{ z \mid |\mathrm{Re}\, z| < \ln 4 \} \tag{3.46}$$

Hence for any $u \in D(0,\alpha)$

$$\rho[0,u,D(0,1)] \geq \rho[0,g(u),\Lambda] \tag{3.47}$$

$$\operatorname{arctanh} \alpha \geq \operatorname{arctanh} \left| \tan \frac{\pi}{4 \ln 4} h(u) \right| \tag{3.48}$$

$$g(u) \in \frac{4 \ln 4}{\pi} \arctan D(0,\alpha) \tag{3.49}$$

which implies (3.33,34).

------

4. Miscellanous Results Regarding Geodesics.

In this section we will prove some properties of the geodesics.

Lemma 4.1: For any $u, v \in \Omega$

$$\frac{F(v,\Omega)}{F(u,\Omega)} \le \sup_{z \in \partial\Omega} \left|\frac{u-z}{v-z}\right|^2 \tag{4.1}$$

Proof: Let $\lambda = \sqrt{\sup(\;'')}$. Then

$$|u-z| \le \lambda |v-z| \tag{4.2}$$

for all $z \in \partial\Omega$. The region of z's satisfying (4.2) is connected and contains $\partial\Omega, u$ but not v (unless $u = v$) so (4.2) holds for all $z \in C \setminus \Omega$. Equation (3.23) can be modified by distributing the lines of charge in $\hat{C} \setminus \Omega : \frac{1}{Im(u,\Omega)} + u$ instead of $\Lambda = Im(u,\Omega)$:

$$V(\Lambda) = \inf_{\mu \in T(\hat{C} \setminus \Omega)} \iint_{(\hat{C} \setminus \Omega)^2} -\ln\left|\frac{1}{z_1-u} - \frac{1}{z_2-u}\right| d\mu(z_1)d\mu(z_2) \tag{4.3}$$

Obviously

$$-\ln\left|\frac{1}{z_1-u} - \frac{1}{z_2-u}\right| = \ln\frac{|z_1-u|\cdot|z_2-u|}{|z_1-z_2|} \tag{4.4}$$

thus

$$V(\Lambda) \leq V[\mathrm{Int}(v,\Omega)] + 2\ln\lambda \tag{4.5}$$

which proves (4.1).

<u>Theorem</u> <u>4.2</u>: Suppose that $\Omega_1 \subset \Omega$ and $f(u,v,\Omega_1)$ can be analytically continued to a univalent (one to one) function of $v \in \Omega$. Then Ω_1 is strongly convex in the hyperbolic geometry of Ω : for any $u,v \in \Omega_1^b$ either

$$\Gamma(u,v,\Omega) \subset \Omega_1 \tag{4.6}$$

or

$$\Gamma(u,v,\Omega) \subset \partial\Omega_1 \tag{4.7}$$

$$\partial\Omega \subset \partial\Omega_1 \tag{4.8}$$

<u>Proof</u>: Clearly we can assume that

$$u, v \in \tilde{\partial}\Omega_1 \tag{4.9}$$

The extended $f(\cdot, v, \Omega_1)$ conformally maps Ω onto some domain, Ω_1 onto $D(\infty)$, u to 0 and v to ∞. Thus we can assume

$$\Omega_1 = D(\infty) \ , \ u = 0 \ , \ v = \infty \tag{4.10}$$

Define

$$\tilde{\Gamma} = (\,|Re| + i\,Im\,)\ \Gamma(0, \infty, \Omega) \tag{4.11}$$

Lemma 4.1 implies that

$$F(\tilde{\Gamma}, \Omega) \le F(\Gamma, \Omega) \tag{4.12}$$

Clearly

$$|d\tilde{\Gamma}| = |d\Gamma| \tag{4.13}$$

so

$$\int_{\tilde{\Gamma}} F(\tilde{z}, \Omega)\,|d\tilde{z}| \le \int_{\Gamma} F(z, \Omega)\,|dz| = \rho(0, \infty, \Omega) \tag{4.14}$$

but $\Gamma(0, \infty, \Omega)$ uniquely minimizes the integral among all curves in $S(0,\infty,\Omega)$ so $\tilde{\Gamma}=\Gamma$

$$\operatorname{Re}\Gamma \geq 0 \tag{4.15}$$

The integral will diverge iff 0 or ∞ are in $\partial\Omega$ but then we take neighbors.

For any $w\in\Gamma(0,\infty,\Omega)$, $\operatorname{Re} w \geq 0$ so

$$D(\infty) \subset \Omega - w \tag{4.16}$$

so

$$\operatorname{Re}[\Gamma(w,\infty,\Omega) - w] \geq 0 \tag{4.17}$$

thus $\operatorname{Re}\Gamma(0,\infty,\Omega)$ is monotonically nondecreasing. If (4.6) is violated a part of $\Gamma(0,\infty)$ is a straight line in $\partial D(\infty)$. It is always true that $f[\Gamma(0,\infty),0]$ is a straight line so the Schwarz reflection principle implies that all $\Gamma(0,\infty)\subset\partial D(\infty)$ and $-D(\infty)\subset\Omega$ which proves (4.7,8).

The univalence reqiment cannot be weakened even to $\partial_u f(u,w,\Omega_1)\neq 0$ for all $u\in\Omega$. For example take

$$\{z \mid \tfrac{\pi}{4} < \operatorname{Arg} z < \tfrac{\pi}{2}\} \subset D(\infty) \tag{4.18}$$

Theorem 4.2 is intended to confine geodesics but it has some spinoffs about univalent continuability. For example:

Corollary 4.3: Suppose that $\Omega \subset D(0,a)$ and $f(u,w,\Omega)$ can be analytically continued to a univalent function of all $u \in D(0, 3a)$. Then Ω is convex.

Proof: For any $u, v \in \Omega$ apply Theorem 4.2 to $\Omega \subset \Omega_2$ where is the disc whose center is on the straight line between u and v which contains and is tangent to $D(0,a)$.

The following result shows how a geodesic between boundary points is perturbed when the domain is perturbed.

Theorem 4.4: Suppose that $\Omega_1 \neq \Omega$

$$\tilde{u}, \tilde{v} \in \partial\Omega \cap \partial\Omega_1 \tag{4.19}$$

and $\tilde{u}, \tilde{v}$ are in the same connected component of $\partial\Omega \setminus \Omega_1$ as well as $\partial\Omega_1 \setminus \Omega$:

$$\tilde{u}, \tilde{v} \in W \cap W_1 \tag{4.20}$$

$$W = \mathrm{Con}(\tilde{u}, \partial\Omega \setminus \Omega_1) \tag{4.21}$$

$$W_1 = \mathrm{Con}(\tilde{u}, \partial\Omega_1 \setminus \Omega) \tag{4.22}$$

where $\mathrm{Con}(u, \mathcal{Q})$ denotes the connected componet of $\mathcal{Q}$ which contains u. Then

$$\Gamma(\tilde{u}, \tilde{v}, \Omega_1) \subset \mathrm{Con}[W_1, \Omega_1^b \setminus \Gamma(\tilde{u}, \tilde{v}, \Omega)] \tag{4.23}$$

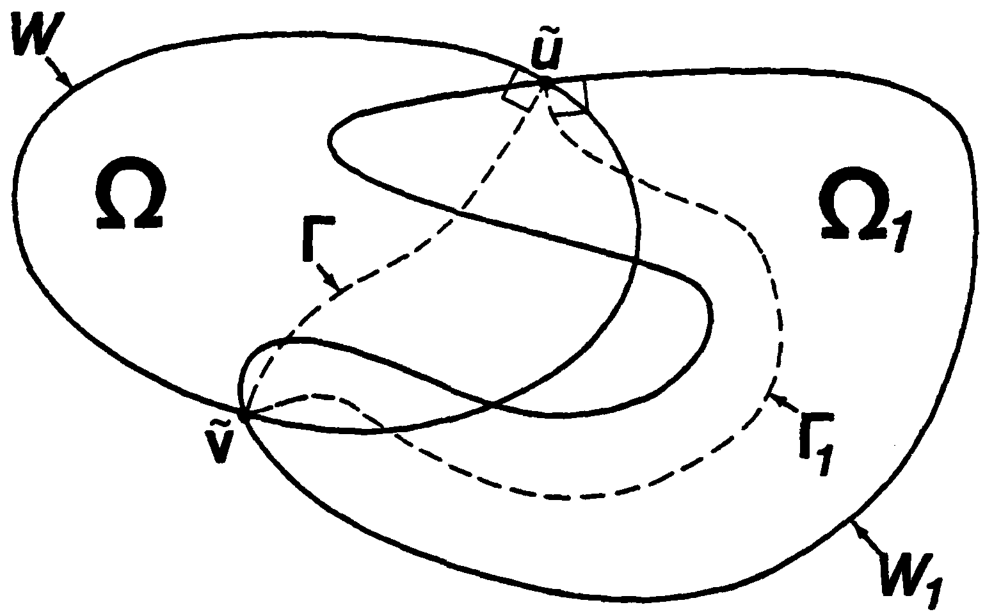
Consult Fig 4.1 .

<u>Proof</u>: When Ω and Ω_1 are disjoint the theorem is trivial. Oterwise define

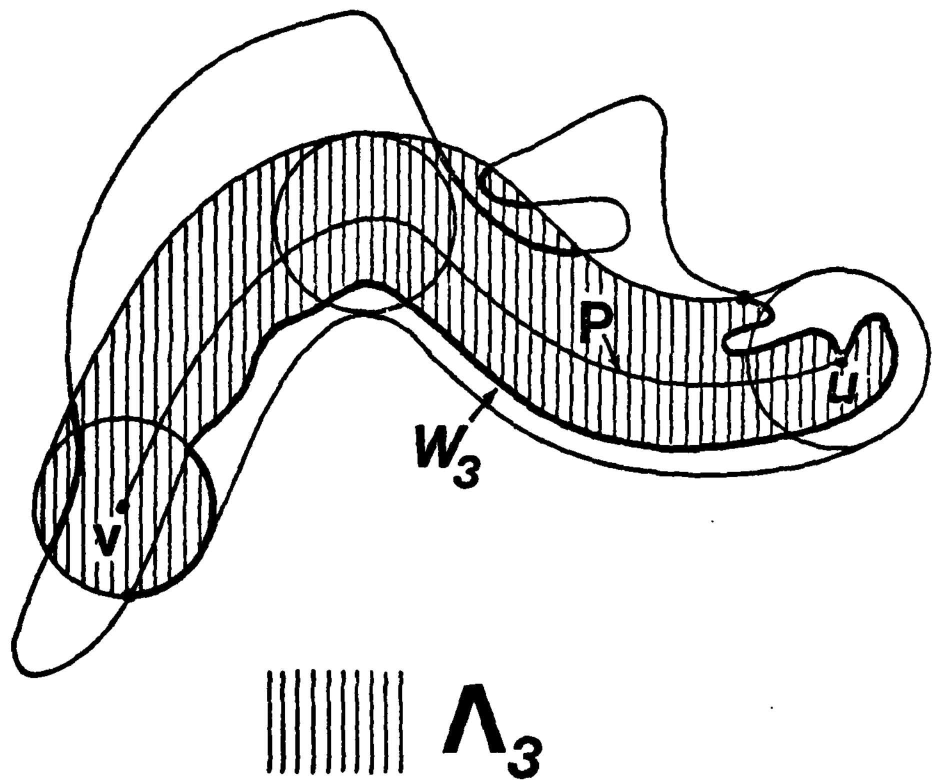
$$\Omega_2 = \mathrm{Con}(\Omega, \mathcal{C} \setminus W^c \setminus W_1^c) \tag{4.24}$$

Clearly $\Omega \cup \Omega_1 \subset \Omega_2$. Let us change $\Omega(0) = \Omega$ to $\Omega(1) = \Omega_2$ continuously and monotonically. A particular scheme will be given in Section 10. Take $\tilde{u}_\varepsilon \in \partial D(\tilde{u}, \varepsilon) \cap W$, $\tilde{v}_\varepsilon \in \partial D(\tilde{v}, \varepsilon) \cap W$ where $\varepsilon \downarrow 0$. For any $0 \leq t \leq 1$ there exists a $\Delta t > 0$ such that

$$\Gamma[\tilde{u}, \tilde{v}, \Omega(t + \Delta t)] \subset \Omega(t) \tag{4.25}$$



W
$\tilde{u}$
Ω
Ω_1
Γ
$\tilde{v}$
Γ_1
W_1


P
u
W3
v
Λ3

Then Theorem 4.2 is applicable to

$$Q_\varepsilon(t+\Delta t) \subset \Omega(t) \tag{4.26}$$

$$Q_\varepsilon(t) = \mathrm{Con}\left[W_1, \Omega_2^b \setminus \Gamma(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \Omega(t))\right] \tag{4.27}$$

Thus

$$\Gamma[\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, \Omega(t)] \subset Q_\varepsilon(t+\Delta t) \tag{4.28}$$

or equivalently

$$Q_\varepsilon(t) \subset Q_\varepsilon(t+\Delta t) \tag{4.29}$$

We have just proven that $Q(0) \subset Q(1)$ which can be written as

$$\mathrm{Con}\left[W_1, \Omega_2^b \setminus \Gamma(\tilde{u}, \tilde{v}, \Omega)\right] \supset \mathrm{Con}\left[W_1, \Omega_2^b \setminus \Gamma(\tilde{u}, \tilde{v}, \Omega_1)\right] \tag{4.30}$$

Similarly

$$\mathrm{Con}\left[W_1, \Omega_2^b \setminus \Gamma(\tilde{u}, \tilde{v}, \Omega_1)\right] \subset \mathrm{Con}\left[W_1, \Omega_2^b \setminus \Gamma(\tilde{u}, \tilde{v}, \Omega_1)\right] \tag{4.31}$$

Formulas (4.30,31) combine into

$$\mathrm{Con}[W_1, \Omega_1^b \setminus \Gamma(\tilde{u}, \vartheta, \Omega_1)] \subset \mathrm{Con}[W_1, \Omega_1^b \setminus \Gamma(\tilde{u}, \tilde{\vartheta}, \Omega)] \qquad (4.32)$$

which implies (4.23).

Formula (4.28) follows directly from (4.26) by a messy computation or a consideration of the second variation of $\rho(\tilde{u}, \tilde{\vartheta})$.

Now let us examine internal endpoints.

__Theorem__ __4.5__: Suppose that $u, \vartheta \in \Omega^b$

$$\Gamma(u, \vartheta, \Omega) \subset \Gamma(\tilde{u}, \tilde{\vartheta}, \Omega) \qquad \tilde{u}, \tilde{\vartheta} \in \partial\Omega \qquad (4.33)$$

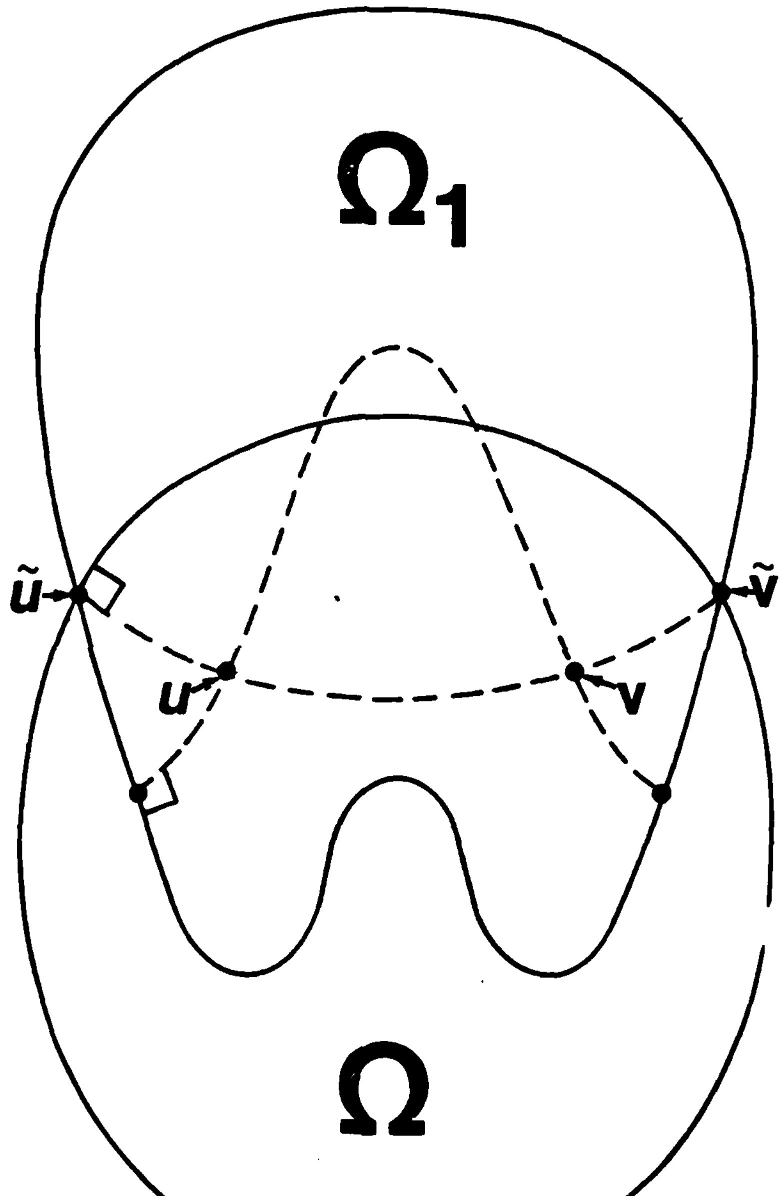
$\tilde{u}, \tilde{\vartheta}, \Omega, \Omega_1$ satisfy all the conditions of Theorem 4.4 plus

$$\Gamma(\tilde{u}, \tilde{\vartheta}, \Omega) \subset \Omega_1 \qquad (4.34)$$

Then

$$\Gamma(u, \vartheta, \Omega_1) \subset \mathrm{Con}[W_1, \Omega_1^b \setminus \Gamma(\tilde{u}, \tilde{\vartheta}, \Omega)] \qquad (4.35)$$

$$C\Gamma(u, \vartheta, \Omega_1) \subset \Omega_1 \setminus \mathrm{Con}[\text{''}] \qquad (4.36)$$



Ω_1
$\tilde{u}$
$\tilde{v}$
u
v
Ω

where $C\Gamma$ is the continuation of Γ beyond its endpoints. See Fig 4.2 .

<u>Proof</u>: Theorem 4.2 is applicable to

$$\mathcal{Q} \subset \Omega \cap \Omega_1 \tag{4.37}$$

$$\mathcal{Q} = \mathcal{C}on[\tilde{\partial}\Omega \setminus W, \Omega^b \setminus \Gamma^c(\tilde{u}, \tilde{v}, \Omega)] \tag{4.38}$$

so

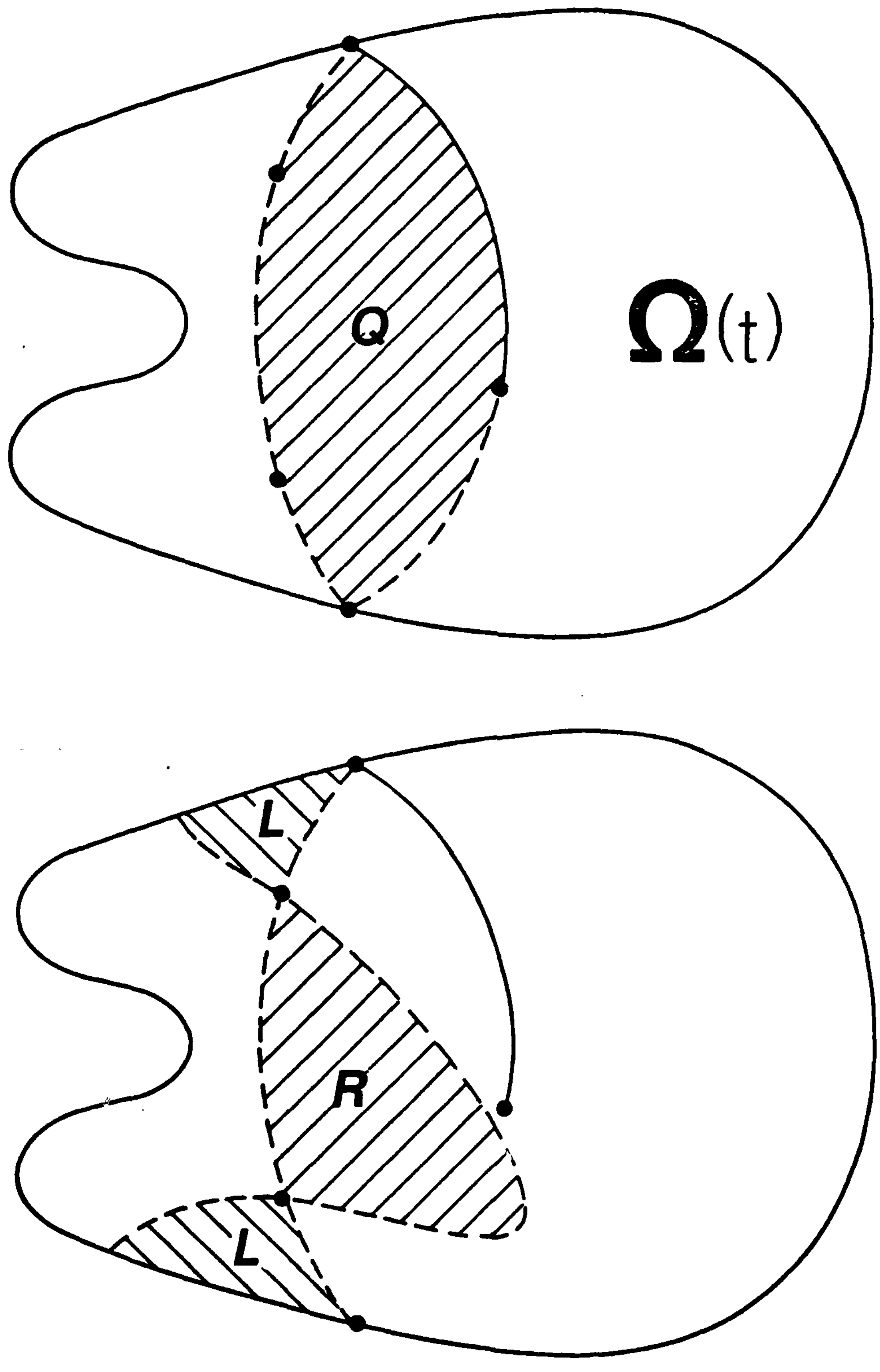
$$\Gamma(u, v, \Omega \cap \Omega_1) \subset \mathcal{Q} \tag{4.39}$$

$$C\Gamma(u, v, \Omega \cap \Omega_1) \subset (\Omega \cap \Omega_1) \setminus \mathcal{Q} \tag{4.40}$$

Let us monotonically transform $\Omega(0) = \Omega \cap \Omega_1$ to $\Omega(1) = \Omega_1$. Define

$$R(t) = \dot{\Omega}_1 \setminus \mathcal{C}on[\tilde{\partial}\Omega_1, \Omega_1^b \setminus \Gamma^c(u, v, \Omega) \setminus \Gamma^c(u, v, \Omega(t))] \tag{4.41}$$

$$\mathcal{X}(t) = \Omega_1 \setminus \mathcal{C}on[W_1, \Omega_1^b \setminus C\Gamma^c(u, v, \Omega) \setminus C\Gamma^c(u, v, \Omega(t))] \tag{4.42}$$


Q
Ω(t)
L
R
L

as in Fig 4.3 . The perturbation argument which we have used to prove Theorem 4.4 shows that

$$R(t+\Delta t) \supset R(t) \tag{4.43}$$

$$\mathcal{L}(t+\Delta t) \supset \mathcal{L}(t) \tag{4.44}$$

We had to use the fact that the boundary endpoints of $C\Gamma(u,v,\Omega(t))$ $\tilde{u}(t), \tilde{v}(t)$ are in $\Omega \cap \partial\Omega_1$. This approach cannot obtain (4.39,40) because when $\Omega(t)$ is shrinking $\tilde{u}(t), \tilde{v}(t)$ may get in the way.

Suppose that (4.35) or (4.36) is violated at "time" but not before. Then in light of (4.39,40,43,44) the total geodesics $T\Gamma[u,v,\Omega(t)]$ and $T\Gamma(u,v,\Omega) = \Gamma(\tilde{u},\tilde{v},\Omega)$ intersect at a point different from u, v

$$w \in [T\Gamma(u,v,\Omega) \cap T\Gamma(u,v,\Omega(t))] \setminus \{u,v\} \tag{4.45}$$

Suppose that (4.36) is violated and $v \in \Gamma[u,w,\Omega(t)]$ (otherwise exchange u, v). Then

$$\Gamma[v,w,\Omega(t)] \subset (\Omega \cap \Omega_1) \setminus Q \tag{4.46}$$

So

$$R(t, v, w) \subset (\Omega \cap \Omega_1) \setminus Q \tag{4.47}$$

where $R(t, v, w)$ denotes the R of formula (4.41) with u, v replaced by v, w. Formulas (4.39,43) imply that when $\Omega \neq \Omega_1$

$$R(t, v, w) \supset R(0, v, w) \not\subset \hat{C} \setminus Q \tag{4.48}$$

which contradicts (4.47). Thus (4.36) holds for all $0 \leq t \leq 1$.
If (4.35) is violated and $v \in \Gamma(u, w, \Omega)$ then

$$C\Gamma[v, w, \Omega(t)] \cap Q \neq \{\ \} \tag{4.49}$$

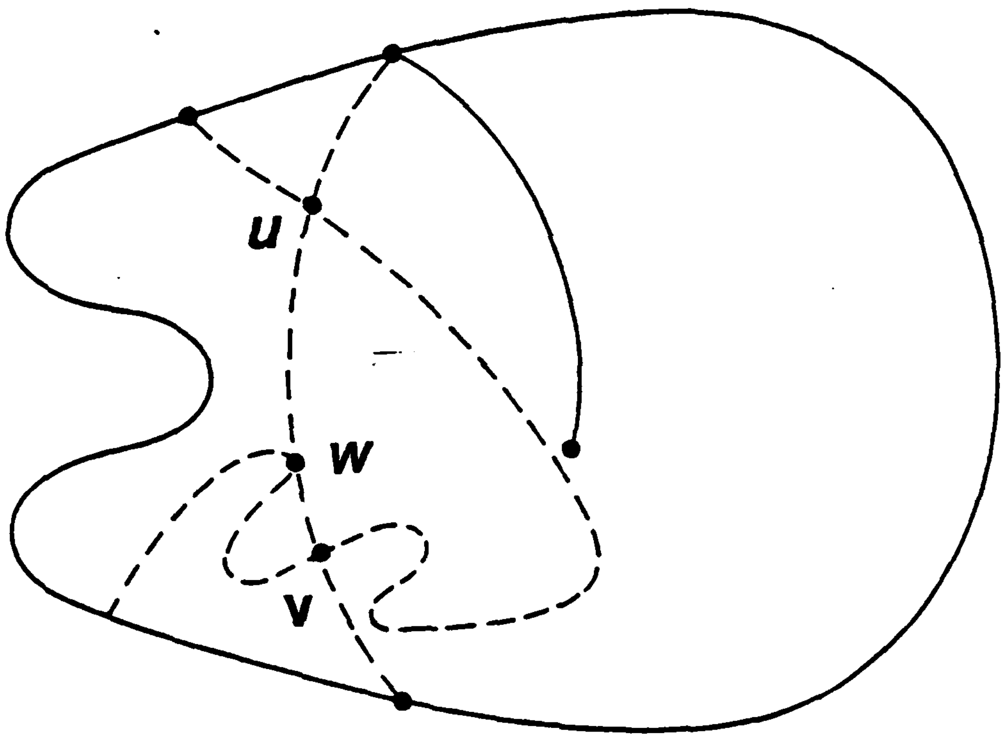
which violates (4.36) for v, w.

The next result helps estimating $\inf \rho[u_1, \Gamma(u_2, u_3)]$, a quantity whose importance will be seen later.

<u>Theorem</u> <u>4.6</u>: For any $u_1, u_2, u_3 \in \Omega$ there exists a $v \in \Omega$ such that

$$\inf \rho[v, \Gamma(u_j, u_k)] < \tfrac{1}{2} \ln(2+\sqrt{3}) \qquad 1 \leq j < k \leq 3 \tag{4.50}$$

Moreover for any $v \in \Omega$


u
w
v

$$\left| \inf \rho[u_1, \rho(u_2, u_3)] - \rho(u_1, v) \right| \leq$$

$$\leq \max_{1 \leq j < k \leq 3} \inf \rho[v, \Gamma(u_j, u_k)] \qquad (4.51)$$

5. Derivatives of Conformal Mappings.

We have already seen that the dependence of the transformation f on the domain Ω is pathological. The correct function to consider is $\ln \partial_u f(u,v,\Omega)$. Formula (1.3) gives its real part and implies

$$\left| \ln|\partial_u f(u,v,\Omega)| + 2\rho(u,v,\Omega) - \ln F(u,\Omega) \right| \leq \ln 4 \tag{5.1}$$

We know how to estimate F and will learn how to estimate ρ but the two terms may cancel each other. When $u \in \partial\Omega$ they are both ∞ . In order to get a useful formula we apply (5.1) at a new center point $w \in \Omega$, subtract it from (5.1) and obtain

$$\left| \ln|\partial_u f(u,\cdot)| \Big|_w^v + 2\rho(v,w) \right| \leq$$

$$\leq 2\left[\rho(u,w) + \rho(w,v) - \rho(u,v)\right] + \ln 16 \tag{5.2}$$

the point w is intended to be near the geodesic connecting u and v. Clearly

$$\rho(u,w) - \rho(w,v) - \rho(u,w) \leq 2 \inf \rho[w, \Gamma(u,v)] \tag{5.3}$$

The distances $|w-u|$, $\inf|w-\partial\Omega|$ should be of the same order as the local length scale of $\partial\Omega$. The term $-2\rho(v,w)$ is interpreted as the "global shape" contribution to $\ln|\partial_n f(u,v)|$ and $\ln|\partial_n f(u,w)|$ as the "local boundary" contribution. That division is fuzzy at best. When Ω is concave near u a cancellation is unavoidable. For example consider a half tube ending in a cone of angle α

$$\Omega = \{ z \mid |\mathrm{Im}\, z| < 1 \,,\ |\mathrm{Arg}\, z| < \tfrac{\alpha}{2} \} \tag{5.4}$$

and let $u \in \partial\Omega$ be near the central corner and v well inside Ω

$$u = \varepsilon e^{i\frac{\alpha}{2}} \quad 0 < \varepsilon \ll 1 \quad , \quad v > 1 \tag{5.5}$$

If w is chosen asccording to the smallest length scale

$$w = \varepsilon \tag{5.6}$$

$$\ln|\partial_n f(u,w)| = \ln\frac{1}{\varepsilon} + O(1) \tag{5.7}$$

$$\rho(v,w) = \frac{\pi}{2} v + \frac{\pi}{2\alpha} \ln\frac{1}{\varepsilon} + O(1) \tag{5.8}$$

It seems we have gone too close. The "correct" w is

$$w = 1 \tag{5.9}$$

$$\ln |\partial_n f(u,w)| = (1-\tfrac{\pi}{\alpha}) \ln \tfrac{1}{\varepsilon} + \mathcal{O}(1) \tag{5.10}$$

$$\rho(v,w) = \frac{\pi}{2} v + \mathcal{O}(1) \tag{5.11}$$

Notice that when the corner is concave $\alpha > \pi$ the local contribution is positive and cancels some of the global contribution. But the local term is only logarithmic in ε.

Theorem 3.4 can be used to estimate $\ln |\partial_n f(u,v)|$. The simplest comparison domain is the inside or outside of a disk:

$$\tilde{\Omega} = \begin{cases} D(0,a) & a > 0 \quad (5.12) \\ \hat{\mathbb{C}} \setminus D(0,-a) & a < 0 \quad (5.13) \end{cases}$$

If u is on the boundary and w on the internal normal at u

$$u \in \partial D(0,|a|) \quad , \quad a\frac{u-w}{u-0} > 0 \tag{5.14}$$

then

$$|\partial_u f(u,w,\tilde{\Omega})| = \frac{2}{|u-w|} - \frac{1}{a} \tag{5.15}$$

Let us consider higher derivatives.

<u>Theorem</u> <u>5.1</u>: For any $u, v, w \in \Omega$, $n \geq 1$

$$\ln \partial_u f(u,\cdot)\Big|_w^v = -2 \ln\left[\frac{1}{f(v,w)} - f(u,w)\right] - -2 \ln \sinh \rho(v,w) + i\,c(v,w) \tag{5.16}$$

$$\frac{\partial_u^n}{n!} \ln \partial_u f(u,\cdot)\Big|_w^v = \sum_{m=1}^{n} 2(m-1)!\left[\frac{1}{f(v,w)} - f(u,w)\right]^{-m} \cdot \sum_{\{\nu_j\}\in I_{n,m}} \prod_{j=1}^{n} \frac{1}{\nu_j!}\left[\frac{\partial_u^j}{j!} f(u,w)\right]^{\nu_j} \tag{5.17}$$

$$I_{n,m} = \left\{ \{\nu_j\}_{j=1}^n \;\middle|\; \nu_j \geq 0,\ \sum_1^n \nu_j = m,\ \sum_1^n j\nu_j = n \right\} \tag{5.18}$$

<u>Proof</u>: Formula (5.16) is obtained by taking the logarithm of (1.12) and inserting (1.4). Formula (5.17) is (5.16)'s n'th derivative.

If we set $w=u$ in (5.17) we obtain the $n \geq 1$ analogue of (1.3). It suffers from the same cancellation problem. Notice that the only v dependence of $\partial_u^n \ln \partial_u f(u,v)$ as expressed by (5.20) comes through $\overline{f(v,v)}$. The term in the square brackets is bounded as follows:

<u>Theorem</u> <u>5.2</u>: For any $u, v \in \Omega$

$$\frac{1}{2} \max\left[\sinh 2\tilde{\rho},\, 1-e^{-2\rho}\right] \leq \left|\frac{1}{\overline{f(v,w)}} - f(u,w)\right|^{-1} \leq$$

$$\leq \frac{1}{2}\left(e^{2\tilde{\rho}}+1\right) \qquad (5.19)$$

where

$$\tilde{\rho} = \inf \rho[w, \Gamma(u,v)] \leq \rho = \rho(v,w) \qquad (5.20)$$

The proof is an elementary exercis involving Lobachevski's geometry.

What have we gained by replacing v with w? The point w is "near" u so the localization theory of section 10 allows us to approximate $\partial_u^n \ln \partial_u f(u,w,\Omega)$ by $\partial_u^n \ln \partial_u f(u,w,\tilde{\Omega})$ where $\tilde{\Omega}$ equals Ω "near" u, w and omits the rest of the domain. Thus $\tilde{\Omega}$ is relatively simple. How to estimate $\partial_u^n f(u)$, $\partial_u^n \ln \partial_u f(u)$?

Theorem **5.3**: For any $w \in \tilde{\Omega}$ the following functions are analytic in $u \in \tilde{\Omega}$

$$g(u,w,\tilde{\Omega}) = \ln \frac{f(u,w,\tilde{\Omega})}{u-w} \tag{5.21}$$

$$h(u,w,\tilde{\Omega}) = i\left[\ln \partial_u f(u,w,\tilde{\Omega}) - g(u,w,\tilde{\Omega})\right] \tag{5.22}$$

and satisfy the boundary conditions

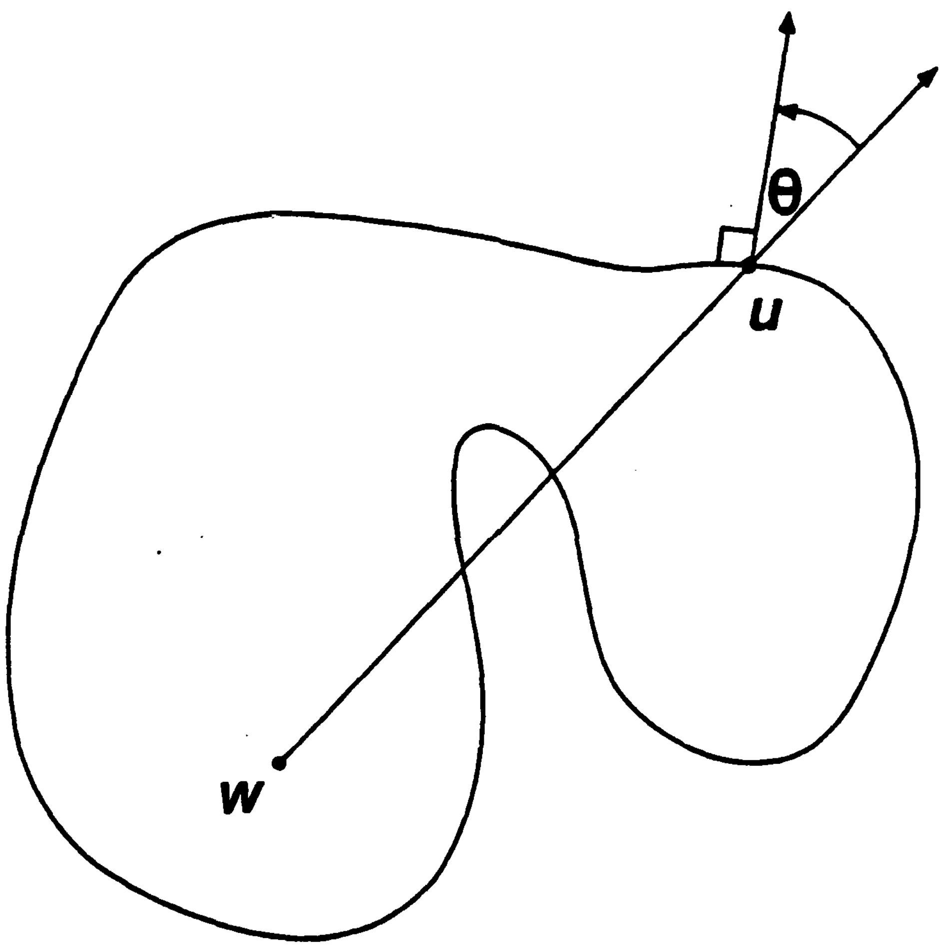
$$\mathrm{Re}\, g(u,w,\tilde{\Omega}) = -\ln|u-w| \qquad u \in \partial\tilde{\Omega} \tag{5.23}$$

$$\mathrm{Re}\, h(u,w,\tilde{\Omega}) = \Theta(u,w,\partial\tilde{\Omega}) \qquad u \in \partial\tilde{\Omega} \tag{5.24}$$

where $\Theta(u,w,\partial\tilde{\Omega})$ is the angle of the outside normal to $\partial\tilde{\Omega}$ at u relative to the straight line between w and u.

See Fig. 5.1 .

Thus we have two basic Dirichlet problems whose solutions and their derivatives provide all the functions we want. In particular notice that for $u \in \partial\Omega$


θ
u
w

$$\ln|\partial_u f(u,w,\tilde{\Omega})| = \ln h(u,w,\tilde{\Omega}) - \ln|u-w| \qquad (5.25)$$

Problems (5.23,24) are the first two in an infinite series whose third involves $\partial\tilde{\Omega}$'s curvature and so on so that the n'th problem involves the first $n-1$ deritatives of f and $\partial\Omega$. The fourth problem onwards can be chosen to be independent of w.

The theory of elliptic boundary value problems is applicable to problems (5.23) and (5.24). It is covered in overwhelming detail in [ ]. Those details shouldn't obscure the fact that the bounds obtained are quite bad. For instance consider the conformal distance $\rho(u,v,\Omega)$. The weakest P.D.E. Theory's restriction on Ω is the cone condition: for any $o \in \Omega$ there exists inside Ω a truncated cone $Q(o)$ starting at o of angle α and length $\mu \sup|\partial\Omega - \partial\Omega|$

$$\forall o \in \Omega \qquad \exists\, Q(o) \subset \Omega \qquad (5.26)$$

This is not enough to bound ρ (since bottlenecks are possible) so we must add a mild smoothness condition such as

$$\left[\sup|\partial\Omega - \partial\Omega| \int_{\partial\Omega} \kappa^2(z,\partial\Omega)\,|dz|\right]^{1/2} \leq a \qquad (5.27)$$

where $\kappa(z, \partial\Omega)$ is the curvature of $\partial\Omega$ at z. The best ρ bound we can expect is

$$\sup_{u,v\in\Omega} \inf \rho[Q(u), Q(v), \Omega] \leq C(\alpha, \mu, a) \tag{5.28}$$

This bound is far from trivial. Notice the subtle way condition (5.27) combines with the cone condition to prevent bottlenecks. However it all seems quite irrelevnat and there is no lower bound. Compare with section 9. A possible basic deficiency of the P.D.E. approach is that (5.23) and (5.24) are very special Dirichlet problems: their domain and boundary conditions are strongly linked.

Here follows a result which seems well beyond the power of P.D.E. Theory. For any two points w_1, w_2 on an open curve P define $\alpha(w_1, w_2, P)$ to be the change in P's angle between w_1 and w_2:

$$\alpha(w_1, w_2, P) = \operatorname{Arg} \frac{dP(w_1)}{dP(w_2)} \tag{5.29}$$

<u>Theorem</u> <u>5.4</u>: For any $u, v \in \Omega$, $q \in \partial\Omega$

$$\left| \operatorname{Arg} \partial_n f(u, v, \Omega) + \alpha(\tilde{u}, \tilde{v}, \partial\Omega \setminus \{q\}) \right| < 4\pi \tag{5.30}$$

where $\tilde{u}, \tilde{v} \in \partial\Omega$ minimize $|\tilde{u} - u|$, $|\tilde{v} - v|$ respectively and f

is normalized so that $\partial_1 f(v, v, \Omega) > 0$.

Proof:

$$\operatorname{Arg} \partial_u f(u) = \operatorname{Arg} \partial_1 f(\cdot)\Big|_{v}^{\tilde{v}} + \text{''}\Big|_{\tilde{v}}^{\tilde{u}} + \text{''}\Big|_{\tilde{u}}^{u} \tag{5.31}$$

For any $w_1, w_2 \in \Omega$

$$\operatorname{Arg} \partial_1 f(\cdot)\Big|_{w_1}^{w_2} = \tilde{\alpha}(w_1, w_2, \Omega) - \tilde{\alpha}[f(w_1), f(w_2), D(0,1)] \tag{5.32}$$

where

$$\tilde{\alpha}(w_1, w_2, \Omega) = \alpha[w_1, w_2, \Gamma(w_1, w_2, \Omega)] \tag{5.33}$$

For any $z_1, z_2 \in D(0,1)$,

$$\left| \tilde{\alpha}[z_1, z_2, D(0,1)] \right| < \pi \tag{5.34}$$

$$\tilde{\alpha}[0, z_2, D(0,1)] = 0 \tag{5.35}$$

Theorem 4.2 implies that

$$\Gamma(\tilde{u}, u, \Omega) \subset D\left(\frac{\tilde{u}+u}{2}, \left| \frac{\tilde{u}-u}{2} \right| \right) \tag{5.36}$$

and of course

$$\Gamma(\tilde{u},u,\Omega) \perp \partial O(\,''\,) \tag{5.37}$$

so

$$|\tilde{\alpha}(\tilde{u},u,\Omega)| \leq \frac{\pi}{2} \tag{5.38}$$

and a similar result holds for $\tilde{v},v$. Moreover (5.37) implies that

$$\tilde{\alpha}(\tilde{u},\tilde{v},\Omega) = \alpha(\tilde{u},\tilde{v},\tilde{\partial}\Omega\setminus\{q\}) \pm \pi \tag{5.39}$$

where the sign depends on q's location. Combining it all together gives (5.30).

------

6. Extremal Length.

This section presents an alternative to the approach of theorem 1.1 towards the estimation of conformal invariants such as $\rho(u,v)$ and the modified harmonic measure

$$l(W,u,\Omega) = \sin\left[\frac{1}{4}\,\mathrm{Length}\ f(W,u,\Omega)\right] \tag{6.1}$$

For any set S of piecewise continuous curves $\subset \hat{C}$ its extremal length $\lambda(S)$ is defined to be

$$\lambda(S) = \sup_{\eta \geq 0} \frac{\left[\inf\limits_{\Gamma \in S} \int_{\Gamma} \eta(z)\,|dz|\right]^2}{\iint\limits_{\hat{C}} \eta^2(x+iy)\,dx\,dy} \tag{6.2}$$

where the metric scalar functions η are smooth and not identically 0. Clearly $\lambda(S)$ is conformally invariant. It is easy to show that:

<u>Theorem 6.1</u>: If $S_1 \subset S_2$ then:

$$\lambda(S_1) \geq \lambda(S_2) \tag{6.3}$$

For any S_1, S_2

$$\lambda(S_1 \oplus S_2) \geq \lambda(S_1) + \lambda(S_2) \tag{6.4}$$

$$\lambda[S_1 \cup S_2] \geq [\lambda^{-1}(S_1) + \lambda^{-1}(S_2)]^{-1} \tag{6.5}$$

Each $\mathcal{S}$ defines its conjugate

$$\mathcal{S}^* = \{ p.c.\ \text{curve}\ P \mid \forall \Gamma \in \mathcal{S} \quad P \wedge \Gamma \neq \{\ \} \} \tag{6.6}$$

All the $\mathcal{S}$'s which we are going to discuss satisfy

$$\mathcal{S}^{**} = \mathcal{S} \tag{6.7}$$

$$\lambda(\mathcal{S}) \cdot \lambda(\mathcal{S}^*) = 1 \tag{6.8}$$

The latter property is of considerable importance because it enables us to bound $\lambda(\mathcal{S})$ from above as well as below by using a single η

$$\frac{\left[\inf\limits_{\Gamma \in \mathcal{S}} \int_\Gamma \eta(z)|dz|\right]^2}{\iint_\Omega \eta^2(x+iy)\,dx\,dy} \leq \lambda(\mathcal{S}) \leq \frac{\iint_\Omega \eta^2(x+iy)\,dx\,dy}{\left[\inf\limits_{P \in \mathcal{S}^*} \int_P \eta(z)|dz|\right]^2} \tag{6.9}$$

Let $U, V \subset \Omega^c$ be two closed sets. Define $\mathcal{S}(U, V, \Omega)$ to be the set of all the piecewise connected curves connecting U and V in Ω

$$S(U,V,\Omega) = \{ \text{p.c. curve } \Gamma \mid U \cap \mathrm{Con}[V, \Gamma \cap \Omega] \neq \{\ \} \} \qquad (6.10)$$

The conjugate $S^*(U,V,\Omega)$ is the set of all the piecewise connected curve separating U from V in Ω

$$S^*(U,V,\Omega) = \{ \text{p.c. curve } \Gamma \mid U \cap \mathrm{Con}[V, \Omega \setminus P] = \{\ \} \} \qquad (6.11)$$

We will abbreviate

$$\lambda(U,V,\Omega) = \lambda[S(U,V,\Omega)] \qquad (6.12)$$

The standard examples are

$$\lambda[i(0,a),\, b+i(0,a),\, (0,b)+i(0,a)] = \frac{b}{a} \qquad (6.13)$$

$$\lambda[\partial D(0,r),\, \partial D(0,R),\, D(0,R) \setminus D^c(0,r)] = \frac{1}{2\pi} \ln \frac{R}{r} \qquad (6.14)$$

The rectangle (6.13) is the canonical domain for $U, V \subset \partial\Omega$ connected curves. The annulus (6.14) is the canonical domain for Ω doubly connected and $\partial\Omega = U \cup V$ where U, V are connected. These two cases are related because an annulus minus a radius equals the exponential of a rectangle and the missing radius is in $S(U,V)$ and minimizes $\int_\Gamma \eta_c(z)|dz|$ for the critical metric function so its absence doesn't change $\lambda(U,V)$. More generally

$$\lambda[\partial D(0,r), \partial D(0,R), D(0,R) \setminus D^c(0,r) \setminus Q] = \frac{1}{2\pi} \ln \frac{R}{r} \qquad (6.15)$$

$$Q = \bigcup_j e^{i\alpha_j}[a_j, R] \qquad 0 \leq \alpha_j < 2\pi \ , \ r \leq a_j \leq R \qquad (6.16)$$

is the general domain for Ω doubly connected, $\partial\Omega = V \cup W$, V, W connected and $U \subset W$. Situations (6.13), (6.14) and (6.15,16) are illustrated in Fig 6.1 . For more details see [6].

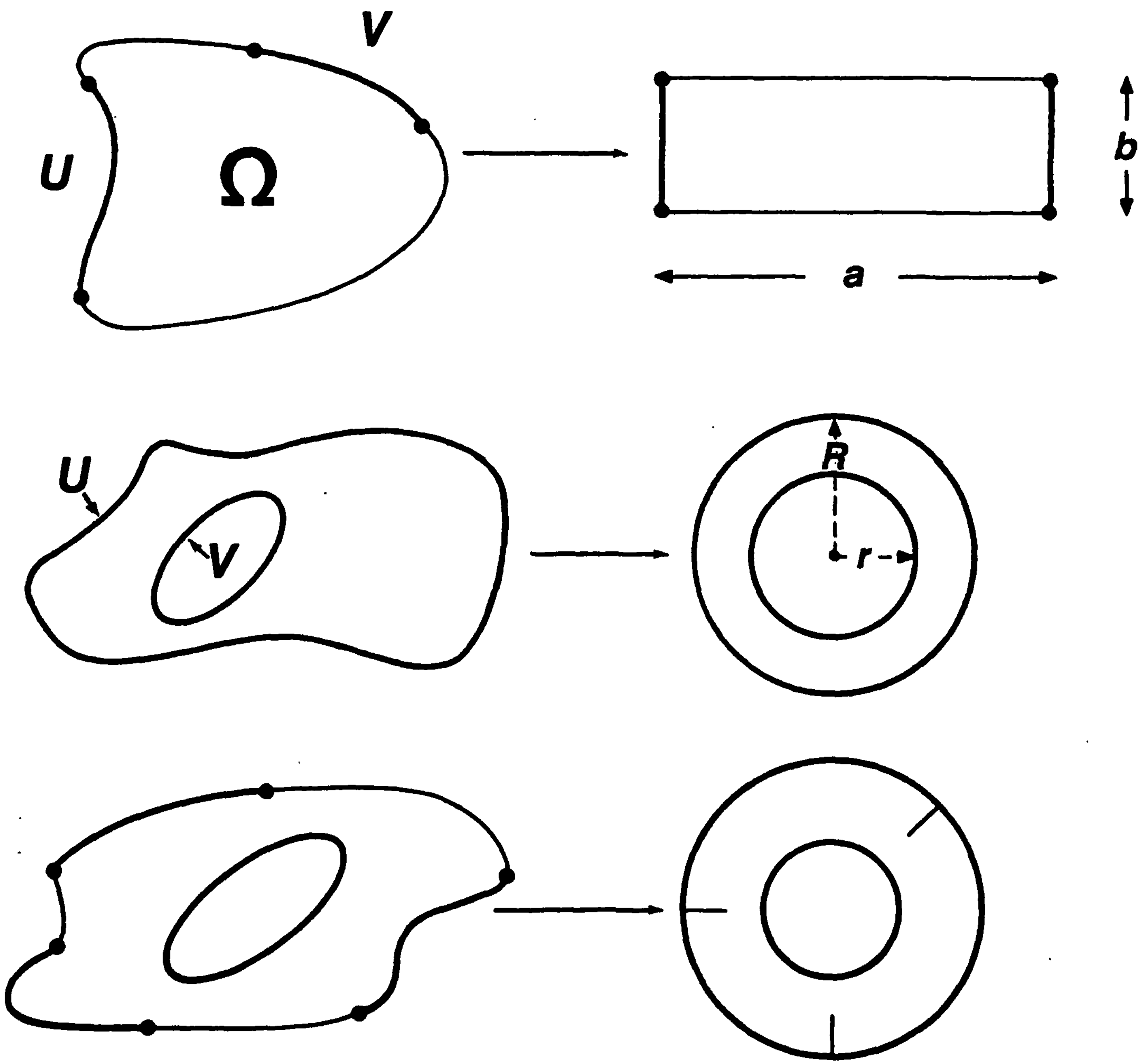
<u>Theorem</u> <u>6.2</u>: Suppose that $\infty \notin \Omega$. Then for any $u, v \in \Omega$ $W \subset \partial\Omega$ connected and $0 < \alpha, \beta \leq \frac{1}{2}$

$$\left| \lambda[D(u,a), \partial\Omega] + \frac{1}{2\pi} \ln[a F(u)] \right| \leq C\alpha \qquad (6.17)$$

$$\left| \lambda[D(u,a), W] - \lambda[D(u,a), \partial\Omega] + \frac{1}{\pi} \ln \ell(W, u) \right| \leq C\alpha \qquad (6.18)$$

$$\Big| \lambda[D(u,a), D(v,b)] - \lambda[D(u,a), \partial\Omega] - \lambda[D(v,b), \partial\Omega] -$$

$$- \frac{1}{\pi} \ln[\tfrac{1}{2} \sinh 2\rho(u,v)] \Big| \leq C(\alpha+\beta) \qquad (6.19)$$


V
U
Ω
b
a
U
V
R
r

$$a = \alpha\, n(u) \quad , \quad b = \beta\, n(v) \tag{6.20}$$

Moreover for general $W \subset \tilde{\partial}\Omega$,

$$\lambda[D(u,a), W] - \lambda[D(u,a), \partial\Omega] \mp \frac{1}{\pi} \ln \ell(W,u) \leq C\alpha \tag{6.21}$$

The limit $\alpha, \beta \downarrow 0$ formulas hold also for $\Omega \ni \infty$.

<u>Proof</u>: Let us start with the $\alpha, \beta \downarrow 0$ case. Then formulas (6.17,18,19,21) are obviously conformally invariant so we can choose a convenient geometry. For (6.17) we choose of course

$$\Omega = D(0,1) \quad , \quad u = 0 \tag{6.22}$$

for (6.18)

$$\Omega = D(0,1) \quad , \quad W = e^{i[-\frac{\omega}{2}, \frac{\omega}{2}]} \tag{6.23}$$

and for (6.19)

$$\Omega = \hat{C} \setminus [-\infty, 0] \quad , \quad v > u = 1 \tag{6.24}$$

the computations are trivial. Formula (6.21) is harder. There we choose situation (6.15,16) with $R = 1$, $r = \varepsilon \downarrow 0$.

Because of (6.15) and (6.18).

$$\lambda[D(u,\varepsilon),W] - \lambda[D(u,\varepsilon),\partial\Omega] \to \frac{1}{2\pi} \ln F(u) \tag{6.25}$$

so we need only show that

$$\ell^2(W,u) \le \frac{1}{F(u)} \tag{6.26}$$

which follows from Theorem 10.2.

The finite α, β case is proven in the same way except that when transforming Ω to a canonical region $\tilde{\Omega}$ one must be able to bound the image of $D[u, \alpha\Omega(u)]$ from above and below and the bounds must agree asymptotically. Those bounds are provided by Theorem 3.6. They immediately translate into λ bounds because

$$U_2, V_2 \subset S^*(U_1, V_1) \Rightarrow \lambda(U_2, V_2) \le \lambda(U_1, V_1) \tag{6.27}$$

------

7. Internal Metrices.

It is time to define a geometric distance between any $u, v \in \Omega$. The Euclidian $|u-v|$ is not satisfactory because it can identify two different $\tilde{\partial}\Omega$ points, or even Ω points for multisheeted domains. The natural candidate is

$$d(u,v,\Omega) = \min_{Z \in S^c(u,v,\Omega)} \int_Z |dz| \tag{7.1}$$

A minimal curve will be denoted by $Z(u,v,\Omega)$.

Theorem 7.1: For any $u, v \in \Omega \setminus \infty$ $Z(u,v,\Omega)$ exists and is unique.

Proof: Existence is well known. Let $Z(u,v,\Omega)$ be a minimal curve. For any $z \in Z$ there exists an $\varepsilon > 0$ such that either $z \in \Omega$ and $Z \cap D(z,\varepsilon)$ is a straight line or $z \in \tilde{\partial}\Omega$ and $Z \cap D(z,\varepsilon)$ is concave relative to Ω. For almost all $z \in Z$ a tangent exists. Define $W(z)$ to be the largest interval of the normal line at z which is connected to u in Ω:

$$W(z) = \mathrm{Con}^0[z, \Omega \cap dZ|_z \cdot (-i\infty, i\infty)] \tag{7.2}$$

Define

$$U(z) = \mathrm{Con}[u, \Omega \setminus W(z)] \qquad (7.3)$$

The local concavity of Z implies that as z tends from u to v $U(z)$ increases:

$$z_2 \in Z(z_1, v, \Omega) \subset Z(u, v, \Omega) \Rightarrow U(z_1) \subset U(z_2) \qquad (7.4)$$

It is easy to show that $\infty \notin \Omega$ implies $v \notin U(z)$ so

$$W(z) \in S^*(u, v, \Omega) \qquad (7.5)$$

Suppose that $P \in S(u, v, \Omega)$. Because of (7.5), for any $z \in Z(u,v,\Omega)$ we can define $\tilde{P}(z)$ as a point in

$$\tilde{P}(z) \in P \cap W^c(z) \qquad (7.6)$$

so that $\tilde{P}$ is a piecewise continuous. Formula (7.4) implies

$$z_1 \neq z_2 \Rightarrow \tilde{P}(z_1) \neq \tilde{P}(z_2) \quad . \qquad (7.7)$$

and because of the local concavity

$$\left| \frac{d\tilde{P}(z)}{dz} \right| \geq 1 \qquad (7.8)$$

thus

$$\text{Length } P \geq d(u,v,\Omega) \tag{7.9}$$

Moreover for equality to hold $\widetilde{P}$ must be parallel to Z at Z's straight parts and identical to Z at the strongly concave parts. That implies that P is identical to Z.

The d disks are

$$D_d(u,a,\Omega) = \{ z \in \Omega \mid d(u,z,\Omega) < a \} \tag{7.10}$$

They are strongly convex relative to Ω.

<u>Theorem</u> <u>7.2</u>: For any $u, v \in D_d^c(w,a,\Omega)$,

$$Z(u,v,\Omega) \subset D_d(w,a,\Omega) \cup [\partial D_d(\;) \setminus \Omega] \tag{7.11}$$

<u>Proof</u>: Suppose that $p \in Z(u,v)$. Parametrize $Z_u = Z(w,u)$, $Z_v = Z(w,v)$ by $0 \leq t \leq 1$. For each $0 \leq t \leq 1$ define $P(t)$ by

$$P(t) \in Z[Z_u(t), Z_v(t)] \tag{7.12}$$

$$d[P(t), Z_u(t)] = d[Z_u(t), Z_v(t)] \frac{d(p,u)}{d(p,v)} \tag{7.13}$$

It is easy to show by a polygonal approximation that

$$|\partial_t P(t)| \leq \frac{d(p,v)}{d(u,v)} |\partial_t z_u(t)| + \frac{d(p,u)}{d(u,v)} |\partial_t z_v(t)| \tag{7.14}$$

and formula (7.14) integrates to

$$d(w,p) \leq \frac{d(p,v)}{d(u,v)} d(w,u) + \frac{d(p,u)}{d(u,v)} d(w,v) \tag{7.15}$$

Equality can hold only when $u \in Z^c(w,v)$ or $v \in Z^c(w,u)$ because otherwise (7.14) is a strict inequality near $t=0$.

Inside Ω, ∂D_d's curvature is limited. In particular

<u>Lemma 7.3</u>: For any $v \in \Omega$, $a > 0$, $u \in D_d(v,a,\Omega)$ there exists a $w \in \partial D(u,\tilde{\alpha})$ such that

$$D(u,\tilde{\alpha}) \cap D(w,\tilde{\alpha}) \subset D_d(v,a,\Omega) \tag{7.16}$$

where

$$\tilde{\alpha} = \min[a, r(u,\Omega)] \tag{7.17}$$

<u>Proof</u>: Define

$$\{w\} = Z(u,v,\Omega) \cap \partial D(u,\tilde{a}) \tag{7.18}$$

then for any $z \in D(u,\tilde{a}) \cap D(w,\tilde{a})$

$$d(v,z) \le d(v,w) + d(w,z) = d(v,u) - \tilde{a} + d(w,z) \le a \tag{7.19}$$

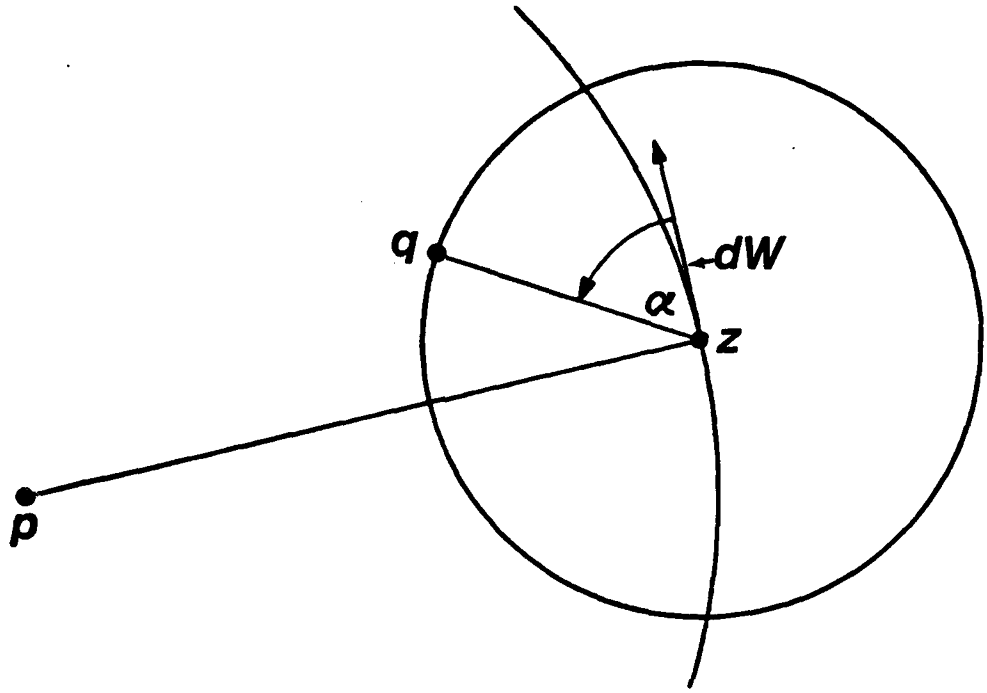
<u>Theorem</u> <u>7.4</u>: For any $v \in \Omega \not\ni \infty$, $a \geq r(v,\Omega)$, $u \in \Omega \cap \tilde{\partial} D_d(v,a,\Omega)$ there exists a curve W such that

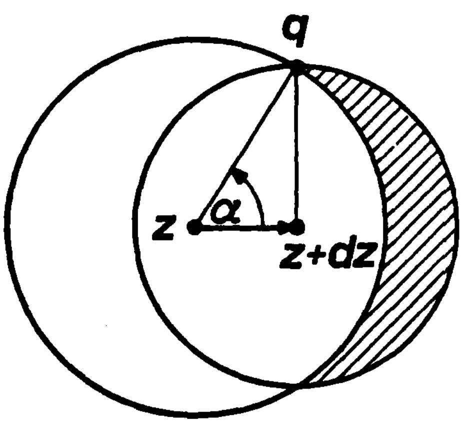
$$W \subset \tilde{\partial} D_d(v,a,\Omega) \ , \quad W \in S(u, \tilde{\partial}\Omega, \Omega) \tag{7.20}$$

$$1 \le \frac{\text{Length } W}{\inf d[u, \partial D_d(v,a,\Omega) \setminus \Omega]} < 17 \tag{7.21}$$

<u>Proof</u>: We will assume $\tilde{\partial}\Omega$ to be smooth. For any $z \in \partial D_d(v,a,\Omega) \cap \Omega$ define $p(z)$ to be the first $Z(z,v,\Omega)$ point in $\tilde{\partial}\Omega$ if any, or v if none

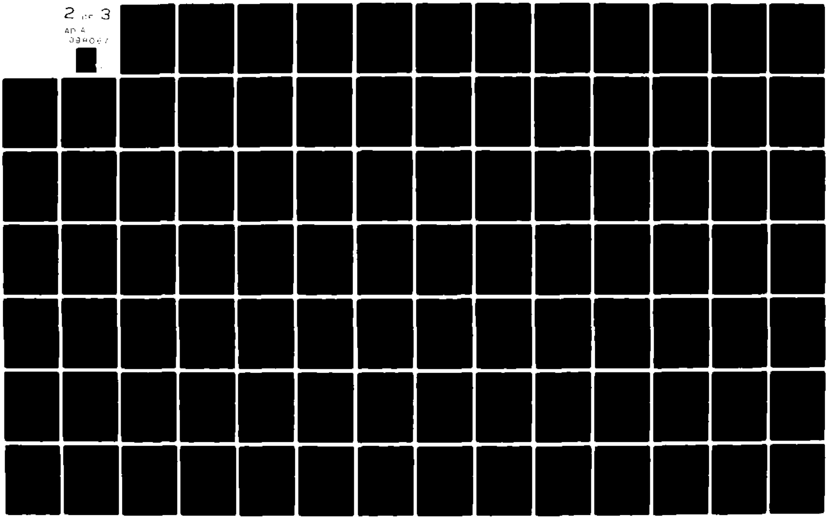
$$p(z) \in [Z(z,v,\Omega) \cap \tilde{\partial}\Omega] \cup \{v\} \ . \tag{7.22}$$


q
dW
α
z
p


q
z
α
z+dz

AD-A098 067 MASSACHUSETTS INST OF TECH CAMBRIDGE DEPT OF MATHEMATICS F/G 12/1
THEORETICAL AND NUMERICAL ANALYSIS OF CONFORMAL MAPPING.(U)
JAN 81 M DUBINER AFOSR-77-3405
UNCLASSIFIED AFOSR-TR-81-0392 NL

2 of 3
AD A
098067



$$(z, p(z)) \subset \Omega \tag{7.23}$$

Define

$$\mu(z) = \inf d[z, \partial D_d(v,a,\Omega) \setminus \Omega] \tag{7.24}$$

$$q(z) \in \partial D_d(v,a,\Omega) \setminus \Omega \quad , \quad d[z, q(z), \Omega] = \mu(z) \tag{7.25}$$

Theorem 7.2 implies that

$$(z, q(z)) \subset D_d(v,a,\Omega) \quad , \quad \mu(z) = |q(z) - z| \tag{7.26}$$

If $\mu(z) \leq a$ then

$$\frac{q(z)-z}{p(z)-z} \notin (0,\infty) \tag{7.27}$$

Otherwise $p(z) = q(z) \neq a$ so $(z, p(z))$ is tangent to $\tilde{\partial}\Omega$ at $p(z)$. Then either $d[v, q(z), \Omega] < a$ so $(z, q(z))$ is perpendicular to $\tilde{\partial}\Omega$ at $q(z)$ which contradicts the tangency or $d[v, q(z), \Omega] = a$ so $a = d(z,v) = d[z, q(z)] + d[q(z), z] > a$.

Suppose that

$$\mu(u) \leq a \tag{7.28}$$

Define W to be the curve which connects u to $\partial\Omega$ in $\partial D_d \setminus \Omega$ and starts in the general direction of $q(u)-u$

$$\operatorname{Re} \frac{dW|_u}{q(u)-u} > 0 \tag{7.29}$$

The curve $\partial D_d \setminus \Omega$ is perpendicular to $(q(u), u)$ at $q(u)$ so (7.27) for $z = u$ implies that (7.29) uniquely determines W's direction. For any $z \in W$ define

$$\alpha(z) = -\operatorname{Arg} \frac{dW|_z}{q(z)-z} \tag{7.30}$$

Denote $dz = dW|_z$. Clearly

$$\alpha(z) \geq 0 \tag{7.31}$$

$$\mu(z+dz) \leq |z+dz-q(z)| =$$

$$= \mu(z) - \cos\alpha(z) \cdot dz + O[(dz)^2] \tag{7.32}$$

$$\frac{d\mu(\mathfrak{z})}{d\mathfrak{z}} \leq -\cos\alpha(\mathfrak{z}) \tag{7.33}$$

$$q(\mathfrak{z}+d\mathfrak{z}) \in D^c[\mathfrak{z}+d\mathfrak{z}, \mu(\mathfrak{z}+d\mathfrak{z})] \setminus D[\mathfrak{z}, \mu(\mathfrak{z})] \tag{7.34}$$

$$\frac{d\alpha(\mathfrak{z})}{d\mathfrak{z}} \leq \sin\alpha(\mathfrak{z}) \tag{7.35}$$

We claim that for all $\mathfrak{z} \in W$

$$\mu(\mathfrak{z}) \leq a \tag{7.36}$$

$$\alpha(\mathfrak{z}) < \frac{\pi}{2} \tag{7.37}$$

Formulas (7.36,37) hold initially at $\mathfrak{z} = \mathfrak{u}$. Let $\mathfrak{z} \in W$ be the first violator. It can not violate (7.36) because (7.33) implies that $\mu(\mathfrak{z})$ is monotonically decreasing up to $\mathfrak{z}$. The function $q(\mathfrak{z})$ is not uniquely defined and $\alpha(\mathfrak{z})$ may be discontinuous but (7.35) implies that any jump up to and at $\mathfrak{z}$ decreases α so $\alpha(\mathfrak{z}) = \frac{\pi}{2}$ which contradicts (7.27).

For any $\mathfrak{z}, \mathfrak{w} \in W$

$$z \neq w \Rightarrow (z, q(z)) \wedge (w, q(w)) = \{\ \} \tag{7.38}$$

Suppose that $z \theta w$ i.e. z separates u from w in W. Then the line $(z, q(z))$ separates u from w in Ω so it separates $Z(v, u, \Omega)$ from $(w, q(w))$ in Ω

$$(z, q(z)) \in S^*[v, (w, q(w)), \Omega] \tag{7.39}$$

Thus for any $w_1 \in (w, q(w))$, $Z(v, w_1, \Omega)$ intersects $[z, q(z)]$ at some point z_1 so

$$d(z_1, w_1) = d(v, w_1) - d(v, z_1) \leq a - [a - d(z_1, z)] \tag{7.40}$$

$$d[q(z), w_1] \leq d[q(z), z_1] + d(z_1, z) = \mu(z) \tag{7.41}$$

$$(w, q(w)) \subset D_d[q(z), \mu(z), \Omega] \tag{7.42}$$

Thus

$$I(z) = \int_{w \theta z} \frac{1}{2} \mu(w) \sin \alpha(w)\, dw \leq$$

$$\leq \text{Area} \bigcup_{w \theta z} (w, q(w)) \leq \pi \mu^2(z) \tag{7.43}$$

Clearly

$$\frac{d}{dz}\sqrt{I(z)} = \frac{-\mu(z)}{4\sqrt{I(z)}}\sin\alpha(z) \leq -\frac{1}{4\sqrt{\pi}}\sin\alpha(z) \tag{7.44}$$

$$\int_{w\ominus z} \sin\alpha(w)\,dw \leq 4\sqrt{\pi}\sqrt{I(u)} \leq 4\pi\mu(u) \tag{7.45}$$

Formula (7.33) implies

$$\int_{w\ominus z} \cos\alpha(w)\,dw \leq \mu(u) \tag{7.46}$$

and $\sin\alpha + \cos\alpha \geq 1$ so

$$\text{Length } W \leq (4\pi+1)\mu(u) \tag{7.47}$$

We are still left with the case

$$\mu(u) > a \tag{7.48}$$

Define u_1 to be the endpoint of

$$P = \text{Con}\,[u, \partial D(v,a) \setminus D_d(v,a,\Omega)] \tag{7.49}$$

which is closest to u. Clearly $\mu(u_1)=a$ so we can start W_1 from u_1 as before. Define

$$W = Con[u_1, P^c \setminus \{u\}] \oplus W_1 \tag{7.50}$$

Then

$$Length\, W \leq \pi \mu(u) + (4\pi + 1)a < (5\pi + 1)\mu(u) \tag{7.51}$$

Suppose that $u, v, w \in \Omega$. The u, v bottleneck's width at w is defined to be

$$b(w,u,v,\Omega) = \sup_{P \in S(u,v,\Omega)} \inf d(w,P,\Omega) =$$

$$= \sup\{b > 0 \mid v \in Con[u, \Omega \setminus D_d(w,b,\Omega)]\} \tag{7.52}$$

<u>Theorem</u> <u>7.5</u>: For any $u, v, w \in \Omega$

$$b(w,u,v,\Omega) = \min[B(w,u,v,\Omega), d(w,u,\Omega), d(w,v,\Omega)] \tag{7.53}$$

where

$$B(w,u,v,\Omega) = \max[\inf d(w,W_1,\Omega), \inf d(w,W_2,\Omega)] \tag{7.54}$$

$$\tilde{\partial}\Omega \setminus \{\tilde{u}, \tilde{v}\} = W_1 \cup W_2 \quad \text{connected} \tag{7.55}$$

and $\tilde{u}, \tilde{v} \in \tilde{\partial}\Omega$ minimize $|\tilde{u}-u|$, $|\tilde{v}-v|$ respectively.

<u>Proof:</u> Obviously

$$u, v \notin D_d(w, a, \Omega) \iff a \le \min d(w, \{u, v\}, \Omega) \tag{7.56}$$

Thus all we have to prove is that when (7.56) holds

$$u \in \mathrm{Con}[v, \Omega \setminus D_d(w, a, \Omega)] \iff a < B(w, u, v, \Omega) \tag{7.57}$$

Assume that

$$a \geq B(w, u, v, \Omega) \tag{7.58}$$

Let $w_1 \in W_1^c, w_2 \in W_2^c$ minimize $d(w, w_1, \Omega)$, $d(w, w_2, \Omega)$ respectively. Clearly

$$Q = Z(w_1, w, \Omega) \oplus Z(w, w_2, \Omega) \subset D_d(w, a, \Omega) \tag{7.59}$$

so it is enough to prove that

$$Q \in S^*(u, v, \Omega) \tag{7.60}$$

Define

$$P = (u, \tilde{u}) \oplus W_1 \oplus (\tilde{v}, v) \in S(u, v, \Omega) \tag{7.61}$$

the line $(u, \tilde{u})$ can not penetrate $Z^c(w, w_j, \Omega)$ because they both minimize the distance from a point (u and w respectively) to W_j. Thus P penetrates Q^c at exactly one point: w_1. Hence P's endpoints u and v are separated by Q.

Assume that

$$a < B(w, u, v, \Omega) \tag{7.62}$$

It implies

$$a < \inf d(w, W_1, \Omega) \tag{7.63}$$

where W_1, W_2 have been switched if necessary. Suppose we could prove the existence of

$$P_1 \in S[u, \tilde{u}, \Omega^c \setminus D_d(w, a, \Omega)] \tag{7.64}$$

Then there also exists $P_2 \in S[\tilde{v}, v, ``]$ so

$$P_1 \oplus W_1 \oplus P_2 \in S[u, v, \Omega^c \setminus D_d(w, a, \Omega)] \tag{7.65}$$

The continuation of $Z(w, u, \Omega)$ intersects $\partial D[u, r(u, \Omega)]$ at a point p

$$p \in \partial D[u, r(u, \Omega)] \quad , \quad u \in Z(w, p, \Omega) \tag{7.66}$$

We claim that if $z \in \partial D[u, r(u, \Omega)]$ tends to p so that $|z - p|$ monotonically decreases then $d(w, z, \Omega)$ monotonically increases. The reason is that a stationary point z is characterized by

$$Z(w, z, \Omega) \perp \partial D[u, r(u, \Omega)] \quad \text{at } z \tag{7.67}$$

in which case either

$$u \in Z(w, z, \Omega) \Rightarrow z = p \tag{7.68}$$

or

$$z \in Z(w, u, \Omega) \Rightarrow z = 2u - p \tag{7.69}$$

Define

$$P_1 = (u, p) \oplus Q \tag{7.70}$$

where Q is the shortest $\partial D[u, \rho(u,\Omega)]$ arc connecting p to $\tilde{u}$. Consult Fig. 7.2. Clearly

$$\inf d(w, P_1, \Omega) \geq \min[d(w,u,\Omega), d(w,\tilde{u},\Omega)] \geq a \tag{7.71}$$

so

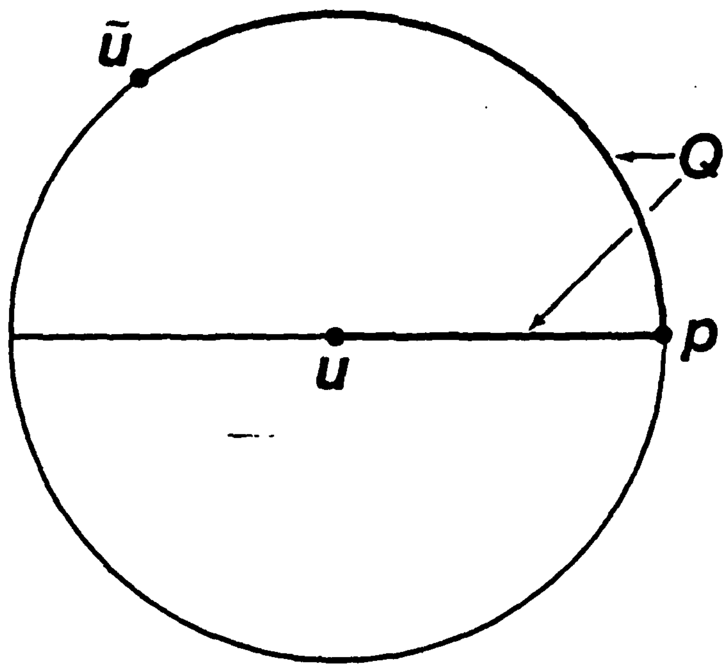
$$P_1 \in S[u, v, D^c(u, \rho(u,\Omega)) \setminus D_d(w, a, \Omega)] \tag{7.72}$$

which implies (7.64).

In Section 9 we will need

<u>Theorem</u> <u>7.6</u>: For any $u, v \in \Omega$, $w \in Z(u,v,\Omega)$ there exists a unit normal $\hat{n}(w,u,v,\Omega)$ to $Z(u,v,\Omega)$ at w such that

$$D[w, b(w,u,v,\Omega)] \cap [w + \hat{n}(w,u,v,\Omega) D(\infty)] \subset \Omega \tag{7.73}$$



ũ
Q
u
p

$$D_d[w, b(w,u,v,\Omega),\Omega] \wedge Z(u,v,\Omega) \subset$$

$$\subset w - \hat{n}(w,u,v,\Omega)\, D(\infty) \qquad (7.74)$$

<u>Proof</u>: Let

$$P \in S(u, v, \Omega^c \setminus D_d(w, b, \Omega)] \qquad (7.75)$$

$$Q = \hat{C} \setminus \mathcal{C}on\,[\hat{C}\setminus\Omega, \hat{C}\setminus P^c \setminus Z(u,v,\Omega)] \subset \Omega \qquad (7.76)$$

Define $\hat{n}(w)$ to be a normal to Z at w in the inside Q direction. It exists because Z is concave relative to Q. Suppose that $\hat{m} \in \hat{n}(w)\, D(\infty)$, $|\hat{m}| = 1$. The line $(w, w + \hat{n}\infty)$ is initially inside Q. Define q to be its last point in Q

$$q \in (w, w + \hat{m}\infty) \wedge \partial Q \qquad (7.77)$$

$$(w, q) \subset Q \qquad (7.78)$$

Formula (7.78) implies that

$$q \in \partial Q \setminus Z^c \subset P \tag{7.79}$$

$$d(w, q, \Omega) \geq b \tag{7.80}$$

$$(w, w + \hat{n} b) \subset (w, q) \subset Q \subset \Omega \tag{7.81}$$

which proves (7.73,74).

______

<u>Lemma</u> <u>7.7</u>: Suppose that $u, v \in \Omega$, $w \in Z(u, v, \Omega)$ and

$$p \in \mathrm{Con}[w, (\Omega \cap [w + \hat{n}(w) D(\infty)]) \cup \{w\}] \cup$$
$$\cup \, \mathrm{Con}[w, (\Omega \cap [w - \hat{n}(w) D(\infty)]) \cup \{w\}] \tag{7.82}$$

Then

$$d[p, Z(u, v, \Omega), \Omega] \geq \left| \mathrm{Re} \frac{p - w}{\hat{n}(w)} \right| \tag{7.83}$$

______

For some purposes the neighborhood $\mathrm{Con}[v, D(v, a) \cap \Omega]$ is preferable to $D_d(v, a, \Omega)$. It is approximately generated by the metric

$$d_0(u,v,\Omega) = \inf\{a>0 \mid \exists\, \sigma\in\hat{\mathbb{C}} \quad u\in \mathrm{Con}[v, D(\sigma,\tfrac{a}{2})\cap\Omega]\} \tag{7.84}$$

Clearly

$$|u-v| \le d_0(u,v,\Omega) \le d(u,v,\Omega) \tag{7.85}$$

The d_0 disks

$$D_0(v,a,\Omega) = \{z\in\Omega \mid d_0(v,z,\Omega) < a\} \tag{7.86}$$

satisfy

$$D_0(v,a,\Omega) \subset \mathrm{Con}[v, D(v,a)\cap\Omega] \subset D_0(v,2a,\Omega) \tag{7.87}$$

By now the reader should have no trouble proving

<u>Theorem</u> <u>7.7</u>: For any $u,v\in\Omega \not\ni \infty$

$$d_0(u,v,\Omega) = \inf\{a>0 \mid \exists\, \sigma\in\hat{\mathbb{C}} \quad Z(u,v,\Omega)\subset D(\sigma,\tfrac{a}{2})\} \tag{7.88}$$

The d_0 bottleneck width b_0 is defined analogously to b.

8. Harmonic Measure Bounds with Applications.

The harmonic measure $\omega(W,\vartheta,\Omega)$ of a curve $W\subset\tilde{\partial}\Omega$ is defined to be the length of its image $\ell(W,\vartheta,\Omega)$ so it can be useo to approximate the image size of some Ω subsets.

Theorem 8.1 is a localization theorem. It implies that the harmonic measure is concentrated at distances of order $\pi(\vartheta,\Omega)$ from the center.

<u>Theorem</u> <u>8.1</u>: For any $\vartheta\in\Omega$, $W\subset\tilde{\partial}\Omega$

$$\ell(W,\vartheta,\Omega)\leq\frac{2}{\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}} \tag{8.1}$$

$$\alpha=\frac{\inf d(W,\vartheta,\Omega)}{\pi(\vartheta,\Omega)} \tag{8.2}$$

the inequality is sharp.

<u>Proof</u>: Normalize

$$\vartheta=0 \quad , \quad \pi(\vartheta,\Omega)=1 \tag{8.3}$$

In light or Theorem 1.2 we can assume that

$$\sup d(\partial\Omega, v) = \infty \tag{8.4}$$

Theorem 6.2 implies that for $\varepsilon \downarrow 0$

$$\lambda[D(0,\varepsilon), W] + \frac{1}{2\pi}\ln[\varepsilon F(0)] + \frac{1}{\pi}\ln \ell(W,0) \leq C\varepsilon \tag{8.5}$$

We choose the metric scalar

$$\eta(u) = \frac{1}{s(|u|)} \qquad u \in \Omega \tag{8.6}$$

$$s(x) = \text{Length}\,[\partial D_d(0,x) \cap \Omega] \tag{8.7}$$

Clearly

$$\iint_{\Omega} \eta^2(x+iy)\,dx\,dy = \int_{\varepsilon}^{\infty} \frac{dx}{s(x)} \tag{8.8}$$

and for each $\Gamma \in S\,[D(0,\varepsilon), W]$

$$\int_{\Gamma} \eta(z)\,|dz| \geq \int_{\varepsilon}^{\infty} \frac{dx}{s(x)} \tag{8.9}$$

Thus

$$\lambda[D(0,\varepsilon), W] \geq \int_{\varepsilon}^{\alpha} \frac{dx}{\ell(x)} \tag{8.10}$$

Obviously

$$\ell(x) \leq 2\pi x \qquad\qquad 0 \leq x \leq 1 \tag{8.11}$$

$$\int_0^x \ell(\rho)\,d\rho = \mathrm{Area}\,[\Omega \cap D_d(0,x)] \leq \pi x^2 \tag{8.12}$$

It is easy to show that (8.11,12) imply

$$\int_{\varepsilon}^{\alpha} \frac{dx}{\ell(x)} \geq \frac{1}{2\pi} \ln \frac{\alpha}{\varepsilon} \tag{8.13}$$

so

$$\lambda[D(0,\varepsilon), W] \geq \frac{1}{2\pi} \ln \frac{\alpha}{\varepsilon} \tag{8.14}$$

Because of (8.4) Theorem 3.3 implies

$$F(0) \geq \frac{1}{4}\left(1 + \frac{1}{\alpha}\right)^2 \tag{8.15}$$

which combines with (8.5) and (8.14) to give (8.1).

With a little extra effort we will prove another result:

Theorem 8.2: For any $\vartheta \in \Omega \not\ni \infty$, $W \in \mathfrak{J}\Omega$

$$\ell(W,\vartheta,\Omega) < \sqrt{32}\, e^{-3/4} \frac{\sqrt{\min(\mu,\alpha)}}{\alpha + \min(\mu,\alpha)} < 4.45 \frac{\sqrt{\mu}}{\alpha} \tag{8.16}$$

$$\alpha = \frac{\inf d(W,\vartheta,\Omega)}{r(0,\Omega)} \tag{8.17}$$

$$\mu = \frac{\inf\{R>0 \mid \exists\, \theta \in \hat{C}\ \ W \subset D(\theta,R)\}}{r(0,\Omega)} \leq \frac{\sup |W-W|}{2\, r(0,\Omega)} \tag{8.18}$$

Proof: Normalize as in (8.3). The case $\mu \geq \alpha$ follows from Theorem 7.1 so we assume

$$\mu < \alpha \tag{8.19}$$

Theorem 6.1 implies

$$\lambda[\partial D(0,\varepsilon), W, \Omega] \geq \lambda[\partial D(0,\varepsilon), U_1, \Lambda_1] + \lambda[W, U_2(\tfrac{\alpha-\mu}{2}), \Lambda_2] \tag{8.20}$$

$$\Lambda_1 = D_d(0, \tfrac{\alpha+\mu}{2}, \Omega) \setminus D(0,\varepsilon) \tag{8.21}$$

$$U_1 = \{u \in \Omega \mid d(u, 0, \Omega) = \frac{\alpha+\mu}{2}\} \tag{8.22}$$

$$\Lambda_2 = \{u \in \Omega \mid \inf d(u, W) < \frac{\alpha-\mu}{2}\} \tag{8.23}$$

$$U_2(x) = \{u \in \Omega \mid \inf d(u, W) = x\} \tag{8.24}$$

As in the previous proof

$$\lambda[D(0,\epsilon), U_1, \Lambda_1] \geq \frac{1}{2\pi} \ln \frac{\alpha+\mu}{2\epsilon} \tag{8.25}$$

$$\lambda[W, U_2, \Lambda_2] \geq \int_{\mu}^{\frac{\alpha+\mu}{2}} \frac{dx}{s(x)} \tag{8.26}$$

$$s(x) = \text{Length } U_2(x-\mu) \tag{8.27}$$

$$\int_{\mu}^{x} s(p)\, dp \leq \pi x^2 \tag{8.28}$$

We have no anologue to (8.11) so we have to be content with the fact that (8.28) implies

$$\int_{\mu}^{\frac{\alpha+\mu}{2}} \frac{dx}{\lambda(x)} \geq \frac{1}{2\pi}\left[\ln\frac{\alpha+\mu}{2\mu} - \left(\ln 2 - \frac{1}{2}\right)\right] \tag{8.29}$$

The combination of (8.5), (3.7), (8.20,25,26,29) results in (8.16).

The next result implies that a geodesic doesn't approach or depart from a boundary point more than a constant times the distance it has to as a curve connecting the geodesic's endpoints in Ω.

<u>Theorem</u> <u>8.3</u>: Suppose that $u, v \in \Omega$. Then any $z \in \Gamma(u,v,\Omega)$ satisfies

$$b(z,u,v,\Omega) \leq (3+\sqrt{8})\, r(z,\Omega) \tag{8.30}$$

and if $\infty \notin \Omega$

$$\inf d[z, Z(u,v,\Omega), \Omega] \leq 19.2\; d_0(u,v,\Omega) \tag{8.31}$$

Inequality (8.30) is sharp.

<u>Corollary</u>: For any $u, v, w \in \Omega$

$$0.14 - 0.16\,\frac{r(w,\Omega)}{b(w,u,v,\Omega)} < \frac{\inf d[w,\Gamma(u,v,\Omega),\Omega]}{b(w,u,v,\Omega)} \leq 1 \tag{8.32}$$

and if $\infty \notin \Omega$

$$1 \leq \frac{\sup d[w,\Gamma(u,v,\Omega),\Omega]}{\max d[w,\{u,v\},\Omega]} < 39 \tag{8.33}$$

Moreover (8.32) and (8.33) hold with d, b replaced by d_0, b_0 .

__Proof__: The goedesic $\Gamma(u,v,\Omega)$ extends to $T\Gamma(u,v,\Omega)$ whose endpoints $\tilde{u},\tilde{v}$ are on the boundary

$$\Gamma(\tilde{u},\tilde{v},\Omega) = T\Gamma(u,v,\Omega) \tag{8.34}$$

For any $\varepsilon > 0$, u and v are connected in

$$\Lambda = \Omega \setminus D_d[z, b(z,u,v,\Omega) - \varepsilon, \Omega] \tag{8.35}$$

Define u_1 to be the first $\Gamma(\tilde{u},u,\Omega)$ point in Λ

$$\Gamma(\tilde{u},u_1,\Omega) \subset \Omega \setminus \Lambda^c \quad , \quad u_1 \in \partial\Lambda \tag{8.36}$$

and similarly v_1 . Define

$$\Omega_1 = \Omega \setminus \Gamma^c(\tilde{u}, u_1, \Omega) \setminus \Gamma^c(v_1, \tilde{v}, \Omega) \tag{8.37}$$

Clearly

$$z \in \Gamma(u, v, \Omega) \subset \Gamma(u_1, v_1, \Omega_1) \tag{8.38}$$

$$b(z, u_1, v_1, \Omega_1) \geq b(z, u, v, \Omega) - \varepsilon \tag{8.39}$$

$$n(z, \Omega_1) \leq n(z, \Omega) \tag{8.40}$$

so it is sufficient to prove (8.30) for u_1, v_1, Ω_1. The points u_1, v_1 split $\partial\Omega_1$ into two connected parts

$$\partial\Omega_1 \setminus \{u_1, v_1\} = W_1 \cup W_2 \quad \text{connected} \tag{8.41}$$

and one of which, say W_1, is at a distance b away from z

$$\inf d(z, W_1, \Omega_1) = b(z, u_1, v_1, \Omega) \tag{8.42}$$

Because of (8.38)

$$\ell(w_1, z) = \frac{1}{\sqrt{2}} \tag{8.43}$$

Thus Theorem 8.1 implies that

$$\frac{1}{\sqrt{2}} \leq \frac{2}{\sqrt{\frac{\ell}{r}} + \sqrt{\frac{r}{\ell}}} \tag{8.44}$$

which proves (8.30).

Now to (8.31). Define u_1 to be the first $\Gamma(\tilde{u}, u, \Omega)$ point which intersects $Z(u, v, \Omega)$, define v_1 similarly, Ω_1 by (8.37) and

$$\Omega_2 = \mathrm{Con}[z, \Omega_1 \setminus Z^c(u, v, \Omega)] \tag{8.45}$$

Clearly

$$d = \inf d[z, Z(u_1, v_1, \Omega_2), \Omega_2] \geq \inf d[z, Z(u, v, \Omega), \Omega] \tag{8.46}$$

and Theorem 7.8 implies that

$$d_0 = d_0(u_1, v_1, \Omega_2) \leq d(u, v, \Omega) \tag{8.47}$$

$$r(z, \Omega_2) \leq d + \frac{1}{\sqrt{2}} d_0 \tag{8.48}$$

Because of Theorem 1.2

$$\ell[z(u_2,v_2,\Omega_2), \mathfrak{z}, \Omega_2] \geq \frac{1}{\sqrt{2}} \tag{8.49}$$

and Theorem 8.2 proves

$$\frac{1}{\sqrt{2}} < 4.45 \frac{\sqrt{\frac{1}{2} d_0 (d + \frac{1}{\sqrt{2}} d_0)}}{d + \frac{1}{2} d_0} \tag{8.50}$$

$$\frac{d}{d_0} < 19.12 \tag{8.51}$$

We have already seen that the conformal map onto can be extremely contracting. However there exists a reasonable bound on its expantion power:

<u>Theorem</u> <u>8.4</u>: For any $u, v, w \in \Omega \ni \infty$

$$1 - |f(u,v,\Omega)| \leq \frac{2}{\sqrt{\frac{d(u,v,\Omega)}{\Lambda(u,\Omega)} + 1} + 1} \tag{8.52}$$

$$|f(u,v,\Omega) - f(w,v,\Omega)| < \frac{18}{\sqrt{\alpha + 1}} \tag{8.53}$$

$$\alpha = \frac{\inf d[v, Z(u,v,\Omega), \Omega]}{d_0(u,w,\Omega)} \tag{8.54}$$

Obviously

$$\alpha + 1 \geq \frac{d_0(u,v,\Omega)}{d_0(u,w,\Omega)} \ , \ \frac{d(u,v,\Omega)}{d(u,w,\Omega)} \tag{8.55}$$

<u>Proof</u>: Clearly

$$\rho(u,v,\Omega) \geq \frac{1}{4} \int_0^{d(u,v,\Omega)} \frac{dx}{x + \Pi(u,\Omega)} = \frac{1}{4} \ln\left[\frac{d(u,v,\Omega)}{\Pi(u,\Omega)} + 1\right] \tag{8.56}$$

which implies (8.52).

It is easy to show that

$$\left| f(u,v,\Omega) - f(w,v,\Omega) \right| \leq \frac{2 \tanh \frac{\rho}{2}}{\cosh^2 \tilde{\rho} + \sinh^2 \tilde{\rho} \tanh^2 \frac{\rho}{2}} \tag{8.57}$$

where

$$\rho = \rho(u,w,\Omega) \tag{8.58}$$

$$\tilde{\rho} = \inf \rho[\Gamma(u,w,\Omega), v, \Omega] \tag{8.59}$$

Let σ minimize $\sup |\sigma - Z(u,v,\Omega)|$. Assume that

$$\Omega(0,\Omega) \le \tfrac{1}{2} d_0(u,w,\Omega) \tag{8.60}$$

Theorem 8.3 implies

$$\tilde{\rho} \geq \frac{1}{4}\int_{19.2}^{\alpha d_0} \frac{dx}{x+\Omega+\frac{1}{2}d_0} \geq \frac{1}{4}\ln\frac{\alpha+1}{20.2} \tag{8.61}$$

so

$$|f(u)-f(w)| \le \frac{4}{e^{2\tilde{\rho}}} \le \frac{4\sqrt{20.2}}{\sqrt{\alpha+1}} \tag{8.62}$$

Now assume that

$$\mu = \frac{\Omega(0,\Omega)}{d_0(u,w,\Omega)} > \frac{1}{2} \tag{8.63}$$

Theorem 4.2 impies

$$\tilde{\rho} \geq \frac{1}{4}\int_{0}^{\alpha d_0} \frac{dx}{x+\Omega+\frac{1}{2}d_0} = \frac{1}{4}\ln\frac{\alpha+\mu+\frac{1}{2}}{\mu+\frac{1}{2}} \tag{8.64}$$

and obviously

$$\rho \le 2\int_{0}^{\frac{1}{2}d_0} \frac{dx}{\Omega-x} = 2\ln\frac{\mu}{\mu-\frac{1}{2}} \;. \tag{8.65}$$

Thus

$$|f(u)-f(w)| \leq \frac{8\,\tanh\frac{\rho}{2}}{e^{2\rho}} \leq \frac{8(\mu-\frac{1}{2})\sqrt{\mu+\frac{1}{2}}}{\mu^2+(\mu-\frac{1}{2})^2}\,\frac{1}{\sqrt{\alpha+\mu+\frac{1}{2}}} \tag{8.66}$$

Theorems 8.1,2 are applicable only in special situations. The Tunnel Lemma is much more versatile but it may give ridiculous numbers.

Lemma 8.5: Suppose that $v \in \Omega$, $u \in \partial\Omega$, $P \in S(u,v,\Omega)$ and $a>0$. Then

$$\ell(\Lambda \cap \partial\Omega, v, \Omega) > 2.25\, e^{-\frac{\pi\,\mathrm{Area}\,\Lambda}{4a^2}} \tag{8.67}$$

where

$$\Lambda = \hat{C} \setminus \mathrm{Com}\left[\infty, \hat{C} \setminus \bigcup_{z \in P} D(z,a)\right] \tag{8.68}$$

Proof: Theorem 3.4 and simple inclusion arguments imply that

$$\ell(\Lambda \cap \partial\Omega, v, \Omega) \geq \ell(\Lambda_1 \cap \partial\Omega, v, \Lambda_1) \geq$$

$$\geq \ell(W_2, v, \Lambda_2) \geq \ell(W_3, v, \Lambda_3) \tag{8.69}$$

where

$$\Lambda_1 = \mathrm{Con}(v, \Lambda \cap \Omega) \tag{8.70}$$

$$W_2 = \mathrm{Con}(u, \Lambda \cap \partial\Omega) \tag{8.71}$$

$$\Lambda_2 = \mathrm{Con}(v, \Lambda \setminus W_2) \tag{8.72}$$

$$W_3 = [W_2 \cap D(v,a)] \cup [\partial D(v,a) \setminus \Lambda_2] \tag{8.73}$$

$$\Lambda_3 = \Lambda_2 \cup D(v,a) \tag{8.74}$$

Consult Fig. 8.1.

Define the metric scalar function

$$\eta(z) = \max\left(1, \frac{a}{\pi |z - v|}\right) \tag{8.75}$$

Take $\varepsilon \downarrow 0$. Clearly

$$\iint\limits_{\Lambda_3 \setminus D(v,\varepsilon)} \eta^2(x+iy)\,dx\,dy = \mathrm{Area}\,\Lambda_3 + \int_{\varepsilon}^{a/\pi} \left[\left(\frac{a}{\pi x}\right)^2 - 1\right] 2\pi x\,dx \leq$$

$$\leq \mathrm{Area}\,\Lambda + \frac{2a^2}{\pi} \ln \frac{a}{\pi\varepsilon} - \frac{a^2}{\pi} \tag{8.76}$$

let

$$Q \in S^*[W_3, \partial D(v,\varepsilon), \Lambda_3 \setminus D(v,\varepsilon)] \tag{8.77}$$

If Q is an open curve it has two endpoints

$$q_1, q_2 \in \partial\Lambda_3 \setminus W_3 \subset \partial\Lambda \tag{8.78}$$

Because of (8.77) Q intersects P at some point p so

$$\int_Q \eta(z)\,|dz| \geq d(p,q_1,\Lambda) + d(p,q_2,\Lambda) \geq 2a \tag{8.79}$$

If Q is closed and of length $\leq 2a$ it is contained in $D(v,a)$ so

$$\int_Q \eta(z)\,|dz| \geq \frac{a}{\pi} \operatorname{Var} \operatorname{Arg}(Q-v) \geq 2a \tag{8.80}$$

Formula (6.9) implies

$$\lambda[W_3, \partial D(v,\varepsilon), \Lambda_3 \setminus D(v,\varepsilon)] \leq \frac{\operatorname{Area}\Lambda}{4a^2} + \frac{1}{2\pi}\ln\frac{a}{\pi\varepsilon} - \frac{1}{4\pi} \tag{8.81}$$

The boundary curve W is connected so Theorem 6.2 applies and proves that

$$\frac{1}{\pi}\ln \ell[W_3, v, \Lambda_3] =$$

$$= -\lambda[W_3, \partial D(v,\varepsilon), \Lambda_3 \setminus D(v,\varepsilon)] - \frac{1}{2\pi}\ln[\varepsilon F(v,\Lambda_3)] \tag{8.82}$$

Obviously

$$F(v,\Lambda_3) \leq \frac{1}{a} \tag{8.83}$$

so

$$\ell(W_3, v, \Lambda_3) \geq \sqrt{\pi}\, e^{-1/4}\, e^{-\frac{\pi \,\mathrm{Area}\,\Lambda}{4a^2}} \tag{8.84}$$

We will use the tunnel Lemma to prove a relatively deep extension of Theorem 3.6. For any $u, v \in \Omega$, $a > 0$ define

$$\delta(u,a,v,\Omega) = \sup \Big|\, |f(D_d(u,a,\Omega), v, \Omega)| - |f(u,v,\Omega)| \,\Big| \tag{8.85}$$

<u>Theorem</u> <u>8.6</u>: For any $u, v \in \Omega \setminus \infty$ there exists a complex ζ $|\zeta| < 1$ such that

$$D[e^{i\xi\delta(u,a,v)} f(u,v), c_1\delta(u,a,v)] \subset$$

$$\subset f[D_d(u,a),v] \subset D[f(u,v), c_2\delta(u,a,v)] \qquad (8.86)$$

and for any $0 < a < d(u,v,\Omega)$ there exists a real $-1 < \xi < 1$ such that

$$D[e^{i\xi\delta(u,\frac{a}{2},v)} f(u,v), c\delta(u,\tfrac{a}{2},v)] \subset$$

$$\subset f[\Omega \setminus \mathrm{Con}(v, \Omega \setminus D_d(u,a)), v] \qquad (8.87)$$

<u>Proof</u>: Let us prove the right side inclusion of (8.86). Obviously

$$f(D_d) \subset D[0, |f(u)| + \delta] \setminus D[0, |f(u)| - \delta] \qquad (8.88)$$

and we can assume that $a < d(u,v)$. Assume

$$a \geq \mathfrak{r}(u,\Omega) \qquad (8.89)$$

It implies

$$f(D_d) \subset D(0,1) \setminus D(0,1-2\delta) \tag{8.90}$$

Let

$$z_1, z_2 \in f(D_d) \quad , \left|\mathrm{Arg}\,\frac{z_2}{z_1}\right| = \mu = \sup\left|\mathrm{Arg}\,\frac{f(D_d)}{f(D_d)}\right| \tag{8.91}$$

$$w \in \mathrm{(1line}\, f\left[f(\Omega \cap \tilde{\partial} D_d) \cap \left\{z \mid \mathrm{Re}\,\frac{z}{\sqrt{z_2 z_1}} > 0\right\}\right] \tag{8.92}$$

Theorem 7.4 provides us with

$$W \subset \tilde{\partial} D_d(u,a,\Omega) \quad , \quad W = S(w, \tilde{\partial}\Omega, \Omega) \tag{8.93}$$

which will be parametrized by its arc length. Define the curve

$$P(s) = W(s) + \frac{\varepsilon}{2}\, r(w,\Omega)\, \partial_s W(s) \qquad 0 \le s \le s_\ell \tag{8.94}$$

where $P(s_\ell)$ is the first point to reach $\tilde{\partial}\Omega$. Lemma 8.5 with $\alpha = \frac{\varepsilon}{2} r(w)$ implies that

$$\ell(\tilde{\partial} E \setminus \tilde{\partial}\Omega, p, E) > 2.25\, e^{-\frac{\pi \,\mathrm{Area}\,\Lambda}{r^2(w)}} \tag{8.95}$$

where

$$E = \mathrm{Com}(0, \Omega \setminus D_\delta^c) \tag{8.96}$$

$$p = P(0) \tag{8.97}$$

$$\Lambda = \hat{C} \setminus \mathrm{Com}[\infty, \hat{C} \setminus \bigcup_{z \in P} D(z, \tfrac{1}{2}\Lambda)] \tag{8.98}$$

Formula (7.40) proves that for some $\sigma \in \Omega$

$$W \subset D[\sigma, \inf d(w, \tilde{\partial}\Omega \setminus \tilde{\partial}E, \Omega)] \tag{8.99}$$

Hence either $\inf d(\,''\,) \leq 2\Lambda(w)$ so

$$\Lambda \subset D[\sigma, 3\Lambda] \tag{8.100}$$

$$\ell(\tilde{\partial}E \setminus \tilde{\partial}\Omega, p, E) > 2.25\, e^{-9\pi^2} \tag{8.101}$$

or $\inf d(\,''\,) > 2\Lambda(w)$ so Theorem 8.1 implies

$$\ell(\,''\,) > 1 - \frac{2}{\sqrt{2} + \frac{1}{\sqrt{2}}} \tag{8.102}$$

By Theorem 3.5

$$|f(p,w,\Omega)| \leq \frac{1}{2} \tag{8.103}$$

so

$$|f(p,\theta) - f(w,\theta)| = \left| f(p,w) \frac{1-|f(\theta,w)|^2}{1-f(p,w)\overline{f(\theta,w)}} \right| \leq$$

$$\leq 1 - |f(\theta,w)|^2 \leq 4\delta \tag{8.104}$$

Formulas (8.92,104) imply that

$$\inf d[f(p), f(\partial E \setminus \partial\Omega), f(E)] \geq |e^{i\frac{\pi}{2}} - 1| - 4\delta \tag{8.105}$$

$$r[f(p), f(E)] \leq 4\delta \tag{8.106}$$

so by Theorem 8.1

$$\ell[f(\partial E \setminus \partial\Omega), f(p), f(E)] \leq \frac{2}{\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}} \tag{8.107}$$

$$\alpha = \frac{2\sin\frac{\pi}{4}}{4\delta} \tag{8.108}$$

which combines with (8.101,102) to bound μ.

We still have to consider $a \leq r(u,\Omega)$. When $a \leq \frac{1}{2} r(u,\Omega)$ Theorem 3.5 implies Theorem 8.5. Thus assume

$$\frac{1}{2} \leq \frac{a}{\Lambda(u,\Omega)} \leq 1 \tag{8.109}$$

For any $w \in \partial D(u,a)$

$$\rho(u,w,\Omega) \geq \frac{1}{4}\int_0^a \frac{dx}{x+\Lambda} \geq \frac{1}{4} \ln \frac{3}{2} \tag{8.110}$$

In prticular $w \in \partial D(u,a) \cap \Gamma(u,\vartheta,\Omega)$ proves

$$\delta(u,a,\vartheta,\Omega) \geq \tanh\left(\frac{1}{4} \ln \frac{3}{2}\right) \cdot \tilde{\delta} \tag{8.111}$$

$$\tilde{\delta} = 1 - \inf \left| f[\partial D(u,a), \vartheta] \right| \tag{8.112}$$

Theorem 4.2 implies that $f[D(u,a)]$ is Lobachevski convex so

$$f[D(u,a)] \subset e^{i\vartheta} Z[D(0,1) \cap D(\infty)] \tag{8.113}$$

$$Z(x) = \frac{x+1-\tilde{\delta}}{1+(1-\tilde{\delta})x} \tag{8.114}$$

The left side inclusion of (8.86) is a corollary of Theorem 3.5 and Lemma 7.3.

The proof of (8.87) has been left to the interested reader. He should at least figure out why $\frac{1}{2}a$ can not be replaced by a (but it can be replaced by αa where $0<\alpha<1$) or ζ by 0. The difference between ζ real and complex is important when one is interested in $\mathcal{J}(\partial E\setminus\Omega, \upsilon, \Omega)$.

9. Estimation of the Conformal Distance and the Location of Geodesics.

In the first part of this section we will estimate $\rho(u,v,\Omega)$ by using formula (3.2) with the F bounds of (3.8).

Theorem 9.1: For any $u, v \in \Omega \setminus \infty$

$$0.01 < \frac{\rho(u,v,\Omega)}{\int\limits_{Z(u,v,\Omega)} \frac{|dz|}{a(z,u,v,\Omega)}} < 15 \tag{9.1}$$

where

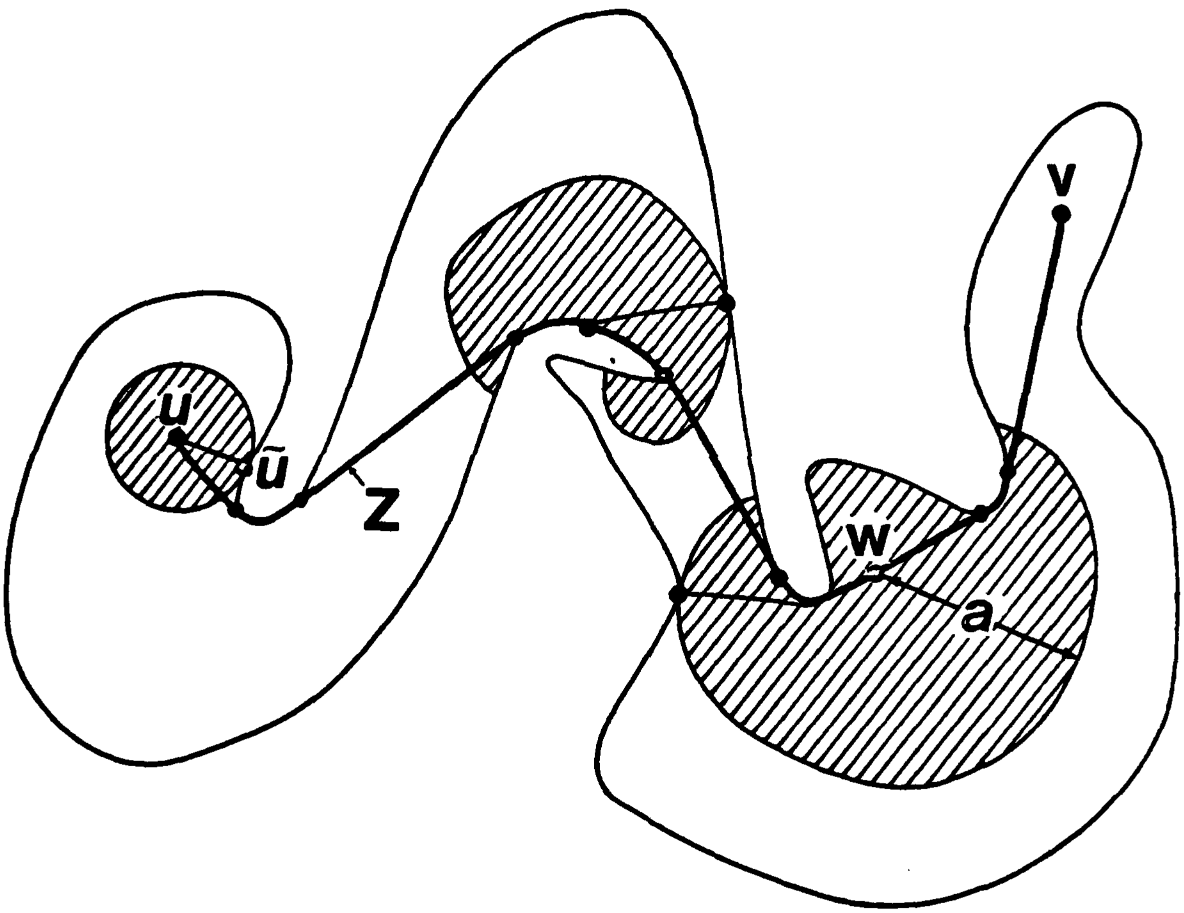
$$a(z,u,v,\Omega) = \mathrm{Max}[b(z,u,v,\Omega),\ r(z,\Omega)] \tag{9.2}$$

Proof: We will prove the upper inequality by construction. Define

$$X(u,v,\Omega) = [(\cdot) + \alpha(\cdot)\hat{n}(\cdot)](Z(u,v)) \tag{9.3}$$

$$\alpha(z) = \tfrac{1}{2}[a(z,u,v,\Omega) - r(z,\Omega)] \tag{9.4}$$

where $\hat{n}(z)$ is provided by Theorem 7.6. The curve $X \in S(u,v,\Omega)$ and is continuous because $\hat{n}$ can flip direction


u
ũ
Z
v
w
a

only when $r < b$. For any $z \in Z$

$$Q = D(z, r) \cup [D(z, b) \cap (z + \hat{n}\, D(\infty))] \subset \Omega \tag{9.5}$$

$$F^{-1}(z + \alpha\hat{n}, \Omega) \geq r(z + \alpha\hat{n}, Q) = \sqrt{r^2(z, \Omega) + \alpha^2} \geq$$

$$\geq \max\left[\frac{1}{\sqrt{5}} a(z, u, v, \Omega), \alpha(z)\right] \tag{9.6}$$

Let us parametrize $Z(u, v, \Omega)$ by its arc length $0 < s < d(u, v, \Omega)$ and for any function $\square$ denote

$$\tilde{\square}(s) = \square([Z(u, v, \Omega)](s)) \tag{9.7}$$

Differention of (9.3) results in

$$\partial_s \tilde{\Sigma}(s) = (1 + |\tilde{\kappa}(s, Z)|\, \tilde{\alpha}(s))\, \hat{\tau}(s) + \partial_s \tilde{\alpha}(s) \cdot \tilde{n}(s) \tag{9.8}$$

where $\kappa(z, Z)$ is Z's curvature at z and $\hat{\tau}(z)$ is the unit tangent at z. We have used the fact that when $z \in \Omega$ $\kappa(z, Z) = 0$ while when $z \in \tilde{\partial}\Omega$ $\hat{n}(z)$ must point inside Ω. Obviously

$$|\partial_s \tilde{\alpha}(s)|, |\partial_s \tilde{n}(s)|, |\partial_s \tilde{\alpha}(s)| \leq 1 \tag{9.9}$$

so

$$|\partial_s \tilde{X}(s)| \leq \sqrt{2} + |\tilde{\kappa}(s)|\, \tilde{\alpha}(s) \tag{9.10}$$

which combines with 9.6 to prove that

$$\rho(u,v) \leq \int_{\tilde{X}} F(z)\,|dz| \leq \int_0^{d(u,v)} \left[\frac{\sqrt{10}}{\tilde{\alpha}(s)} + \tilde{\kappa}(s)\right] ds \tag{9.11}$$

We want to bound the total curvature of $Z(u,v,\Omega)$. First concentrate on the x interval

$$0 \leq s - \tfrac{1}{3}\tilde{\alpha}(s) < x < s + \tfrac{1}{3}\tilde{\alpha}(s) \tag{9.12}$$

There (9.9) implies

$$\tilde{\alpha}(x) \geq \frac{2}{3}\tilde{\alpha}(s) \tag{9.13}$$

$$D_d[Z(s), \tfrac{1}{3}\tilde{\alpha}(s)] \subset D_d[Z(x), \tilde{\alpha}(x)] \tag{9.14}$$

so by Theorem 7.6

$$Z \cap D_d[Z(s), \tfrac{1}{3}\tilde{\alpha}(s)] \subset Z(x) - \tilde{n}(x)\, D(\infty) \tag{9.15}$$

Hence for any two different $x_1 < x_2$ in (9.15) either $\tilde{n}(x_1) \neq \tilde{n}(x_2)$

or $Z([x_1, x_2])$ is a straight line. Thus

$$\int_{s-\frac{2}{3}\tilde{a}(s)}^{s+\frac{2}{3}\tilde{a}(s)} |\tilde{K}(x)|\, dx \le 2\pi \tag{9.16}$$

Also

$$\int_{s-\frac{2}{3}\tilde{a}(s)}^{s+\frac{2}{3}\tilde{a}(s)} \frac{dx}{\tilde{a}(x)} \ge 2 \int_0^{\frac{2}{3}\tilde{a}(s)} \frac{dp}{\tilde{a}(s)+p} = 2 \ln \frac{5}{3} \tag{9.17}$$

We will divide $[0, d(u,v)]$ into intervals of type (9.12). Start from $x_0 = 0$. Assume that we have

$$0 \le x_k \le d - \frac{2}{3}\tilde{a}(d) \tag{9.18}$$

There exist a unique $0 < s_{k+1} < d$ such that

$$s_{k+1} - \frac{2}{3}\tilde{a}(s_{k+1}) = x_k \tag{9.19}$$

and it defines

$$x_{k+1} = s_{k+1} + \frac{2}{3}\tilde{a}(s_{k+1}) \tag{9.20}$$

The procedure stops when (9.18) is violated by x_m.

$$[0, d] = \bigcup_{k=0}^{m-1} (x_k, x_{k+1}] \cup (x_m, d] \tag{9.21}$$

Clearly

$$z((x_m, d]) \subset D^c[\vartheta, \tfrac{4}{3}\rho(\vartheta,\Omega)] \tag{9.22}$$

so $z((x_m, d])$ is a straight line. Summing up (9.16,17) in the intervals $(x_k, x_{k+1}]$ we obtain

$$\int_0^d |\tilde{\kappa}(x)|\, dx \le \frac{\pi}{\ln\frac{4}{3}} \int_0^d \frac{dx}{\tilde{a}(x)} \tag{9.23}$$

which combines with (9.11) to give

$$\rho(u,\vartheta,\Omega) \le \left(\sqrt{10} + \frac{\pi}{\ln\frac{4}{3}}\right) \int_0^d \frac{ds}{\tilde{a}(s)} \tag{9.24}$$

The lower ρ bound is proven by an argument slightly reminiscent of our total curvature bound (9.23). We will obtain $\{x_k\}_{k=0}^{m+1}$ monotonically increasing from $x_0 = 0$ to $x_{m+1} = d(u,\vartheta)$ such that

$$\Delta_k = (x_{k+1} - x_k) - (a_k + a_{k+1}) > 0 \tag{9.25}$$

where

$$a_k = \begin{cases} \tilde{a}(x_k, u, \vartheta, \Omega) & 1 \le k \le m \quad (9.26) \\ 0 & k = 0, m+1 \quad (9.27) \end{cases}$$

Define

$$z_k = Z(x_k) \tag{9.28}$$

$$D_k = \{ z \in \Omega \mid d(z, z_k, \Omega) \le a_k \} \tag{9.29}$$

Formula (9.25) implies that

$$x_n - x_k > a_n + a_k \qquad 0 \le k < n \le m+1 \tag{9.30}$$

$$D_k \cap D_n = \{\ \} \qquad \text{''} \tag{9.31}$$

By definition $\{u\} = D_0$, $\{v\} = D_1$ so (9.31) proves that

$$a_k = b(z_k, u, v) > r(z_k) \qquad 1 \le k \le m \tag{9.32}$$

and formula (7.60) implies

$$S(u, v, \Omega \setminus D_k) = \{\ \} \tag{9.33}$$

$$Q_k = \bigcup_{z \in \partial D_k \cap \partial\Omega} Z^c(z_k, z) \subset D_k \tag{9.34}$$

Thus for any $0 \le j < k < n \le m+1$

$$Z[(0, x_j - a_j)] \oplus S(D_j, D_n, \Omega \setminus Q_k) \oplus Z[(x_n - a_n, d)] \subset$$

$$\subset S(u, v, \Omega \setminus Q_k) = \{\,\} \tag{9.35}$$

$$S(D_j, D_n, \Omega \setminus Q_k) = \{\,\} \tag{9.36}$$

Hence $\Gamma(u, v, \Omega)$ intersects Q_1 at some u_1, $\Gamma(u_1, v, \Omega)$ intersects Q_2 at some u_2 and so on:

$$\rho(u, v) = \sum_{k=0}^{m} \rho(u_k, u_{k+1}) \tag{9.37}$$

$$u_k \in Q_k \tag{9.38}$$

where of course $u_0 = u$, $u_{m+1} = v$. Formulas (9.34,36) imply that

$$d(z_k, u_k) \le a_k - r(u_k) \qquad 1 \le k \le m \tag{9.39}$$

$$d(u_k,u_{k+1}) \geq \Delta_k + r(u_k) + r(u_{k+1}) \qquad 1 \leq k \leq m-1 \tag{9.40}$$

$$\rho(u_k,u_{k+1}) \geq \frac{1}{4}\int_0^{d(u_k,u_{k+1})} \frac{d\rho}{r(u_k)+\rho} \geq \frac{1}{4}\ln\left(2+\frac{\Delta_k + r(u_{k+1})}{r(u_k)}\right) \tag{9.41}$$

By a_k's definition

$$r(u_k) \leq a_k \tag{9.42}$$

and we can exchange u_k, u_{k+1} in (9.41) so

$$\rho(u_k,u_{k+1}) \geq \frac{1}{4}\ln\left[3+\frac{\Delta_k}{\min(a_k,a_{k+1})}\right] \tag{9.43}$$

Similarly one proves

$$\rho(u_0,u_1) \geq \frac{1}{4}\ln\left[2+\frac{}{\min[r(u),a_1]}\right] \tag{9.44}$$

and its $\rho(u_m,u_{m+1})$ analogue. For $m=0$

$$\rho(u,v) \geq \frac{1}{4}\ln\left[1+\frac{d(u,v)}{\min[r(u),r(v)]}\right] \tag{9.45}$$

The sequence $\{x_k\}$ still has to be constructed. We will first construct another sequence $\{t_k\}$. Start from $t_0=0$ and inductively given t_k define t_{k+1} to be the minimal

$t_{k+1} > t_k$ satisfying one of the four conditions

$$\tilde{a}(t_{k+1}) = \lambda\, \tilde{a}(t_k) \qquad \boxplus \qquad (9.46)$$

$$\tilde{a}(t_{k+1}) = \frac{1}{\lambda}\, \tilde{a}(t_k) \qquad \boxminus \qquad (9.47)$$

$$t_{k+1} - t_k = \mu\, \tilde{a}(t_k) \qquad \square \qquad (9.48)$$

$$t_{k+1} = d(u,v) \qquad \boxdot \qquad (9.49)$$

where

$$\mu > \lambda + 1 > 3 \qquad (9.50)$$

are fixed numbers to be chosen later. The symbols to the right of each condition among (9.46-49) denote the type of intervals (t_k, t_{k+1}) satisfying it. Clearly for $\boxtimes = \boxminus, \square, \boxdot$

$$\int_{\boxtimes} \frac{dt}{\tilde{a}(t)} \le \int_0^{1-1/\lambda} \frac{dp}{1-p} + \int_{1-1/\lambda}^{\mu} \lambda\, dp =: \eta = \mu\lambda + \ln\lambda - \lambda - 1 \qquad (9.51)$$

for $\boxplus$

$$\int_{⊞} \frac{d\ell}{\tilde{a}(\ell)} \leq \int_0^{1-2/\lambda} \frac{dp}{1-p} + \int_{1-2/\lambda}^{\mu-(\lambda-2/\lambda)} \lambda \, dp + \int_{\mu-(\lambda-2/\lambda)}^{\mu} \frac{dp}{p+\lambda-\mu} =$$

$$= \phi = \mu\lambda + 3\ln\lambda - \lambda^2 + \lambda - 2 < \eta \tag{9.52}$$

and when $t_1 = d(u,v)$

$$\int_0^d \frac{d\ell}{\tilde{a}(\ell)} \leq \int_0^{\alpha} \min\left(\frac{1}{1-p}, \lambda\right) dp = \begin{cases} \ln \dfrac{1}{1-\alpha} & 0 \leq \alpha \leq 1-\lambda \quad (9.53) \\ \eta - (\mu-\alpha)\lambda & 1-\frac{2}{\lambda} < \alpha < \mu \quad (9.54) \end{cases}$$

$$\alpha = \frac{d(u,v)}{\Lambda(u)} \tag{9.55}$$

Recall

$$[0, d(u,v)] = \cdot ⊠^{m_1} ⊡ \tag{9.56}$$

where each ⊠ can be an interval of any type, · denotes the initial point and multipication and exponentiation denote the union of intervals. The string of symbols is uniquely broken into the following substrings:

$$(\cdot)\, ⊞^n ⊟^k (⊡) \qquad\qquad n, k \geq 1 \tag{9.57}$$

$$(\cdot)\,\boxplus^{n}\,\square\,\boxminus^{k}(\cdot) \qquad n,k \geq 0 \tag{9.58}$$

$$\cdot\,\boxminus^{k}(\boxdot) \qquad k \geq 1 \tag{9.59}$$

$$(\cdot)\,\boxplus^{n}\,\boxdot \qquad n \geq 1 \tag{9.60}$$

$$\cdot\,\boxdot \tag{9.61}$$

where $(\cdot)$ is either $\cdot$ or blank and similarly $(\boxdot)$ is either $\boxdot$ or blank. Each substring forms one $(x_j, x_{j+1}]$ interval. From (9.43-45) we will derive

$$\rho(u_j, u_{j+1}) \geq A_j \tag{9.62}$$

and from (9.51-54)

$$\int_{x_j}^{x_{j+1}} \frac{ds}{a(s)} \leq B_j \tag{9.63}$$

so summing up

$$\frac{\rho(u,v)}{\int_0^{d(u,v)} \frac{ds}{\tilde{a}(s)}} \geq \min_j \frac{A_j}{B_j} \geq \min_\nu E_\nu \tag{9.64}$$

where ν runs over several cases which will be specified. We will consider each of (9.57-61) in turn. The details are unimportant but the reader should understand why our method works for $\mu >> \lambda >> 1$ and why a simpler breakup then (9.57-61) would not do.

I) $\quad (\cdot)\boxplus^n \boxminus^k (\boxdot) \qquad n, k \geq 1$

The t_j's definition does not treat $y_0 = 0$ in any special way so it can be considered as an inside point. We will take no account of the possible $\boxdot$ interval besides adding η to $B(\boxplus^n \boxminus^k)$

$$B[(\cdot)\boxplus^n \boxminus^k \boxdot] = B(\boxplus^n \boxminus^k) + \eta = n\phi + (k+1)\eta \tag{9.65}$$

Let

$$\boxplus^n \boxminus^k = (x_j, x_{j+1}] = (t_{l-n}, t_{l+k}] \tag{9.66}$$

and denote

$$\tilde{a} = \tilde{a}(x_\ell) \tag{9.67}$$

Then

$$a_j = \tilde{a}(x_{\ell-n}) = \lambda^{-n}\tilde{a} \quad , \quad a_{j+1} = \lambda^{-k}\tilde{a} \tag{9.68}$$

and because of (9.9)

$$x_\ell - x_{\ell-n} \geq (1 - \lambda^{-n})\,\tilde{a} \tag{9.69}$$

$$x_{j+1} - x_j \geq (2 - \lambda^{-n} - \lambda^{-k})\,\tilde{a} \tag{9.70}$$

which combines with (9.25) to give

$$\Delta_j \geq 2(1 - \lambda^{-n} - \lambda^{-k})\,\tilde{a} \tag{9.71}$$

$$\Delta_j \geq 2(1 - \tfrac{2}{\lambda})\,\tilde{a} > 0 \tag{9.72}$$

From now till the end of the proof we will denote

$$A = A(\gamma) = \tfrac{1}{4} \ln \gamma \tag{9.73}$$

Case I) breaks into two parts

<u>Ia</u>) $\lambda^{-n} \le \lambda^{-k} \Rightarrow n \ge k$

$$\gamma = 3 + \frac{\Delta}{\lambda^{-n}\tilde{a}} \ge 2\lambda^{n} - 2\lambda^{n-k} + 1 > 2\left(1 - \frac{1}{\lambda}\right)\lambda^{n} \tag{9.74}$$

$$A > \frac{n}{4}\ln\lambda + \ln\left[2\left(1 - \frac{1}{\lambda}\right)\right] \tag{9.75}$$

$$B < n\phi + (n+1)\eta \tag{9.76}$$

$$E = \frac{A}{B} > \frac{\frac{n}{4}\ln\lambda + \ln\left[2\left(1 - \frac{1}{\lambda}\right)\right]}{n\phi + (n+1)\eta} \tag{9.77}$$

This lower bound is a linear fraction in n . It is positive for $1 \le n \le \infty$ so its minimum in that interval is obtained at one of the endpoints

* $$E_1 = E\big|_{n=1} = \frac{\frac{1}{4}\ln(2\lambda - 2)}{\phi + 2\eta} \tag{9.78}$$

* $$E_2 = E\big|_{n=\infty} = \frac{\frac{1}{4}\ln\lambda}{\phi + \eta} \tag{9.79}$$

<u>Ib</u>) $\lambda^{-n} > \lambda^{-k} \Rightarrow n \le k$

$$\gamma = 3 + \frac{\Delta}{\lambda^{-k}\tilde{a}} > 2(\lambda^{k} - \lambda^{k-n}) \geqslant 2(1 - \tfrac{1}{\lambda})\lambda^{k} \tag{9.80}$$

and the rest is identical to case Ia) with n replaced by k.

<u>II</u>) $(\cdot)\boxplus^{n}\square\boxminus^{k}(\boxdot)$ $\qquad n, k \geqslant 0$

$$\boxplus^{n}\square\boxminus^{k} = (x_{l-n}, x_{l+k+1}] \tag{9.81}$$

Denote

$$\tilde{a} = a_{l} \tag{9.82}$$

$$\alpha = \frac{a_{l+1}}{a_{l}} \quad , \quad \frac{1}{\lambda} < \alpha < \lambda \tag{9.83}$$

Then

$$\Delta \geqslant [1 - \lambda^{-n} + \mu + \alpha(1 - \lambda^{-k}) - \lambda^{-n} - \alpha\lambda^{-k}]\tilde{a} =$$

$$= [\mu + 1 - 2\lambda^{-n} + \alpha(1 - 2\lambda^{-k})]\tilde{a} \tag{9.84}$$

$$\Delta \geqslant (\mu - 1 - \alpha)\tilde{a} > (\mu - \lambda - 1)\tilde{a} > 0 \tag{9.85}$$

IIa) $\lambda^{-n} \leq \alpha \lambda^{-k} \Rightarrow n \geq k$

$$\gamma = 3 + \frac{\Delta}{\lambda^{-n} \hat{a}} > (\mu + 1)\lambda^{n} + \alpha(\lambda^{n} - 2\lambda^{n-k}) \geq (\mu + 1 - \lambda)\lambda^{n} \tag{9.86}$$

$$E = \frac{\frac{1}{4} \ln[(\mu + 1 - \lambda)\lambda^{n}]}{n\phi + (n+2)\eta} \tag{9.87}$$

$$* \quad E_3 = \frac{\frac{1}{4} \ln(\mu - \lambda + 1)}{2\eta} \tag{9.88}$$

IIb) $\lambda^{-n} \geq \alpha \lambda^{-k} \Rightarrow n \leq k$

$$\gamma = 3 + \frac{\Delta}{\alpha \lambda^{-k} \hat{a}} > \left(\frac{\mu + 1 - 2\lambda^{-n}}{\alpha} + 1\right)\lambda^{k} > \left(\frac{\mu - 1}{\lambda} + 1\right)\lambda^{k} \tag{9.89}$$

$$E = \frac{\frac{1}{4} \ln[(\mu + \lambda - 1)\lambda^{k-1}]}{k\phi + (k+2)\eta} \tag{9.90}$$

$$* \quad E_4 = \frac{\frac{1}{4} \ln \frac{\mu + \lambda - 1}{\lambda}}{2\eta} \tag{9.91}$$

III) $\cdot\ \boxminus^{k}(\boxminus) \qquad k \geq 1$

$$\cdot\ \boxminus^{k} = [0, x_k] \tag{9.92}$$

$$\tilde{a} = r(u,\Omega) \tag{9.93}$$

$$\Delta \geq (1-\lambda^{-k}-\lambda^{k})\,\tilde{a} > 0 \tag{9.94}$$

and by (9.44)

$$\gamma = 2 + \frac{\Delta}{\lambda^{-k}\tilde{a}} > \lambda^{k} \tag{9.95}$$

$$E = \frac{\frac{1}{2}\ln \lambda^{k}}{(k+1)\,\eta} \tag{9.96}$$

* $$E_5 = \frac{\frac{1}{2}\ln \lambda}{2\eta} \tag{9.97}$$

<u>IV</u>) $(\cdot)\boxplus^{n}\boxdot$ $\qquad n \geq 1$

This case is very similar to III)

$$E = \frac{\frac{1}{2}\ln \lambda^{n}}{n\phi + \eta} \tag{9.98}$$

* $$E_6 = \frac{\frac{1}{2}\ln \lambda}{\phi + \eta} \tag{9.99}$$

v) $\cdot \boxed{\cdot}$

$$\tilde{a} = n(u, \Omega) \tag{9.100}$$

$$\alpha = \frac{d(u,v)}{\tilde{a}} < \mu \tag{9.101}$$

$$E = \begin{cases} \dfrac{\frac{1}{4}\ln(1+\alpha)}{\ln \frac{1}{1-\alpha}} & 0 < \alpha \leq 1 - \frac{1}{\lambda} \quad (9.102) \\[2ex] \dfrac{\frac{1}{4}\ln(1+\alpha)}{\eta - (\mu - \alpha)\lambda} & 1 - \frac{1}{\lambda} < \alpha < \mu \quad (9.103) \end{cases}$$

Obviously (9.102) is minimized at $\alpha = 0$

$$* \qquad E_\eta = \frac{1}{4} \tag{9.104}$$

and it is easy to prove that (9.103) is minimized at $\alpha = \mu$

$$* \qquad E_\eta = \frac{\frac{1}{4}\ln(\mu+1)}{\eta} \tag{9.105}$$

Now insert

$$\lambda = 2 + \varepsilon \qquad \varepsilon \downarrow 0 \tag{9.106}$$

$$\mu = 3 + 2\varepsilon \tag{9.107}$$

which results in

$$\min_{\nu} E_{\nu} = \frac{\ln^2 2}{48 + 20 \ln 2} \tag{9.108}$$

Formula (9.) is the simplest we could devise. A better estimate is provided by

$$F^{-1}(z + \alpha\hat{n}, \Omega) \approx \frac{2}{\pi}[a(z) + \Lambda(z)] \tag{9.109}$$

$$\rho(u,v) \approx \frac{\pi}{4} \int_{z} \frac{\sqrt{4 + \chi^2(z)[a(z) - \Lambda(z)]^2 + \left[\left|\frac{d}{dz}\right|[a(z) - \Lambda(z)]\right]^2}}{a(z) + \Lambda(z)} |dz| \tag{9.110}$$

Formula (9.110) is guaranteed to be correct up to a constant factor for every domain, is asymptotically correct for slender domains and is hopefully reasonably accurate in general.

Besides the conformal distance ρ we will be interested in $\inf \rho[W, \Gamma(u,v)]$ and thus in the geodesics. It is well known that minimizing $\int_{\Gamma} F |dz|$ is much easier than finding the minimal path of integration . Theorem 9.1's proof's approach seems powerless to confront that problem but Theorem 8.3 comes to the rescue.

__Theorem 9.7__: For any $u, v \in \Omega \ni \infty$

$$\sup_{w \in \Gamma(u,v)} \inf_{x \in \Sigma(u,v)} \rho(x,w) \leq \sup_{x \in \Sigma(u,v)} \inf_{w \in \Gamma(u,v)} \rho(x,w) < 24 \quad (9.111)$$

__Proof__: Let $x \in \Sigma$ be generated by the center of coardinates

$$x = 0 + \alpha(0)\hat{n}(0) \quad (9.112)$$

and also normalize

$$b(0,u,v,\Omega) = \hat{n}(\,) = 1 \quad (9.113)$$

The basic idea of the proof follows. Suppose we are given a curve $P(t)$ $0 \leq t \leq l$ starting from $P(0) = \alpha$. Define the monotonically increasing domains

$$\Lambda(t) = \bigcup_{0 \leq t' \leq t} D[P(t'), \frac{1}{\sqrt{5}}] \quad (9.114)$$

By formula (9.6)

$$\Lambda(0) = D(\alpha, \tfrac{1}{\sqrt{\varepsilon}}) \subset \Omega \tag{9.115}$$

Define s to be the first $s > 0$ such that $\Lambda^c(s) \not\subset \Omega$:

$$\Lambda(s) \subset \Omega \tag{9.116}$$

and there exists

$$q \in \partial\Lambda(s) \cap \partial\Omega \tag{9.117}$$

Define

$$p = P(s) \tag{9.118}$$

$$\tilde{P} = P \oplus (p, q) \tag{9.119}$$

When $\Lambda^c(l) \subset \Omega$ define $s = l$, $\tilde{P} = P$. Suppose we have proven the existence of

$$w \in \Gamma(u, v) \cap \tilde{P} \quad . \tag{9.120}$$

Then either $w \in P$ so that

$$\rho(\alpha, w) \le \rho(\alpha, p) \tag{9.121}$$

$$\rho(\alpha, p) \le \sqrt{5}\ \text{Length}\ P \tag{9.122}$$

or $w \in (p, q)$ so that

$$\rho(\alpha, w) \le \rho(\alpha, p) + \rho(p, w) \tag{9.123}$$

$$\rho(p, w) \le \rho\left[0, |w-p|, D(0, \tfrac{1}{\sqrt{5}})\right] =$$

$$= \frac{1}{2} \ln\left(\frac{2}{\sqrt{5}\,|w-q|} - 1\right) \tag{9.124}$$

and one has to bound $|w-q|$ from below. When (9.120) will be established we will assume that $w \in (p, q)$.

Define the curves

$$P_1(t) = \begin{cases} \alpha + (\mu - \alpha)t & 0 \le t \le 1 \quad (9.125) \\ \mu e^{i(t-1)} & 1 < t \le 1 + 2\pi - \theta \quad (9.126) \end{cases}$$

$$P_2(t) = \overline{P_1}(t) \tag{9.127}$$

$$P_{-1}(t) = \begin{cases} \alpha - (\mu+\alpha)t & 0 \le t \le 1 \quad (9.128) \\ -\mu e^{i(t-1)} & 1 < t \le 1 + 2\pi - \theta \quad (9.129) \end{cases}$$

$$P_{-2}(t) = \overline{P_{-1}(t)} \tag{9.130}$$

where

$$\mu > 1 + \frac{1}{\sqrt{5}} \tag{9.131}$$

$$\arcsin \frac{1}{\sqrt{5}\,\mu} < \theta < \frac{\pi}{2} - \arcsin \frac{1}{\sqrt{5}\,\mu} \tag{9.132}$$

Denote

$$s_+ = \min(s_1, s_2) \tag{9.133}$$

$$s_- = \min(s_{-1}, s_{-2}) \tag{9.134}$$

$$\sigma = \begin{cases} + & s_+ \geq s_- \\ - & s_+ < s_- \end{cases} \qquad \begin{matrix} (9.135) \\ (9.136) \end{matrix}$$

and similarly use the notation P_+, P_- etc. We will consider three cases.

I) $s_\sigma < 1+\theta$

Clearly

$$\tilde{P}_+ \oplus \tilde{P}_- \in S^*(u,v) \tag{9.137}$$

so there exists

$$\omega \in \Gamma(u,v) \cap (\tilde{P}_+^c \cup \tilde{P}_-^c) \tag{9.138}$$

Denote the $\tilde{P}_\pm^c$ containing ω by $\tilde{P}^c$ etc. Theorem 8.3 and Lemma 7.7 imply that

$$(3+\sqrt{8})\,|\omega - q| \geq b(\omega,u,v) \geq |R_\ell\, \omega| \tag{9.139}$$

Obviously

$$|\mathrm{Re}\, w| \geq \begin{cases} |q| & q \in [-\mu, 0] \cup [\alpha, \mu] \quad (9.140) \\ \mu \cos\theta - \frac{1}{\sqrt{5}} & q \in e^{i[-\theta,\theta]} \cup e^{i[\pi-\theta,\pi+\theta]} \quad (9.141) \end{cases}$$

$$b(w,u,v) \geq 1 - |q| - \frac{1}{\sqrt{5}} \qquad (9.142)$$

and by Theorem 7.6

$$\mathrm{Re}\, w \geq \frac{2}{\sqrt{5}} \qquad P = P_+ \,,\ q \in [\alpha, \mu] \qquad (9.143)$$

Altogether

$$(3+\sqrt{7})\,|w-q| \geq \min\left(\frac{1}{2} - \frac{1}{2\sqrt{5}},\ \mu\cos\theta - \frac{1}{\sqrt{5}}\right) \qquad (9.144)$$

$$\text{Length}\ P \leq (1+\theta)\mu + \frac{1}{2} \qquad (9.145)$$

__II)__ $s_\sigma > 1+\theta \geq s_{-\sigma}$.

Clearly

$$\mathrm{Con}(\alpha, \Omega \cap (\sigma\infty, \alpha]) \oplus \tilde{P}_{-\sigma} \in S^*(u,v) \tag{9.146}$$

so there exists either

$$w \in \Gamma(u,v) \cap (\tilde{P}^c_{-\sigma} \cup [\sigma\mu, \alpha]) \tag{9.147}$$

or

$$y \in \Gamma(u,v) \cap \mathrm{Con}[\sigma\mu, \Omega \cap \sigma\cdot(\mu, +\infty)] \tag{9.148}$$

Situation (9.147) has already been treated. For (9.148) notice that there always exists

$$\tilde{u} \in \Gamma(u,v) \cap D_d(0,1) \tag{9.149}$$

Hence there exists

$$w \in \Gamma(\tilde{u}, y) \cap (\tilde{P}^c_{\sigma 1} \cup \tilde{P}^c_{\sigma 2}) \tag{9.150}$$

Theorem 8.3 implies

$$(3+\sqrt{8})\,|w-q| \geq b(w,\tilde{u},y) \geq \inf d\,[w, Z(\tilde{u},0)\ominus(0,y)] \geq$$

$$\geq \min\left[\,|w|-1\,,\ \begin{cases} |\operatorname{Im} w| & \sigma \operatorname{Re} w \geq 0 \\ |w| & \sigma \operatorname{Re} w < 0 \end{cases}\right] \geq$$

$$\geq \min\left(\mu - 1 - \frac{1}{\sqrt{5}}\,,\ \mu \sin\theta - \frac{1}{\sqrt{5}}\right) \tag{9.151}$$

and obviously

$$\text{Length } P \leq (1+2\pi-\theta)\,\mu \cdot \frac{1}{2} \tag{9.152}$$

<u>III</u>) $\quad \ell_{-\sigma} > 1+\theta$

Clearly

$$\operatorname{Con}\,[0, \Omega \cap (-\infty,\infty)] \in S^*(u,v) \tag{9.153}$$

thus there exists either

$$w \in \Gamma(u,v) \cap [-\mu,\mu] \tag{9.154}$$

or

$$y \in \Gamma(u,v) \cap [(-\infty,-\mu) \cup (\mu,+\infty)] \tag{9.155}$$

$$(0,y) \in \Omega \tag{9.156}$$

$$w \in \Gamma(\tilde{u},y) \cap (\tilde{P}^c_{\operatorname{sign} y} \cup \tilde{P}^c_{2\operatorname{sign} y}) \tag{9.157}$$

Situation (9.154) is trivial and (9.155-157) has been covered in II).

Inserting $\mu = 1.486$, $\theta = 1.265$ into (9.121-124), (9.144,145), (9.151,152) results in the right hand side inequality of (9.111). The left follows from

$$w \in \Gamma(w_1,w_2) \Rightarrow \rho(x,w) \le \max[\rho(x,w_1),\rho(x,w_2)] \tag{9.158}$$

<u>Theorem</u> <u>9.3</u>: For any $u,v \in \Omega \not\ni \infty$ the curves $\Gamma(u,v,\Omega)$ and $\Sigma(u,v,\Omega)$ can be parametrized by $0 \le t \le 1$ so that

$$\rho(\Sigma[t], \Gamma[t]) \leq c \tag{9.159}$$

$$\frac{1}{c} \leq \frac{\Omega(\Gamma[t], \Omega)}{\Omega(\Sigma[t], \Omega)} \leq c \tag{9.160}$$

$$\left| \ln \frac{\partial_t \Gamma[t]}{\partial_t \Sigma[t]} \right| \leq c \tag{9.161}$$

<u>Sketch</u> <u>of</u> <u>proof</u>: Normalize $d(u, v, \Omega) = 1$ and parametrize by Σ's arc length. Divide the interval $[0,1]$ into

$$(0,1] = \bigcup_{k=0}^{m} (t_k, t_{k+1}] \tag{9.162}$$

as in (9.46-49). Use Theorem 9.2 to determine $\{w_k\}_{k=1}^{m}$ and for each $0 \leq t \leq 1$ define $\Gamma[t]$ to be the piecewise linear interpolation in Γ's arc length of

$$\Gamma[\;] : \{t_k\}_{k=0}^{m+1} \longrightarrow \{u\} \oplus \{w_k\}_{k=1}^{m} \oplus \{v\} \tag{9.163}$$

Formula (9.159) holds at the t_k's and implies (9.160) there, but a much better c is obtainable by the construction in the proof of Theorem 9.2. When $\lambda, \mu >> 1$ formulas (9.159,160) are easily extendable to all $[0,1]$ and the real part of

$\ln \frac{\partial_t \Gamma[t]}{\partial_t \Sigma[t]}$ is bounded. The imaginary part is bounded at the t_k's by ω_k's construction and Theorem 5.4, and the bound is extended to all t by formula (9.16). The case $m=0$ requires special consideration but notice that Theorem 9.2's proof shows that

$$\rho(x,w) \le c\,\rho(u,v) \tag{9.164}$$

and the rest is similar.

------

10. Perturbation and Localization Theory.

The goal of this section is to estimate the change in conformal mapping related functions induced by changing the domain Ω to $\tilde{\Omega}$, with as little relience as possible on detailed structure. Most of this section is dedicated to the simplest conformal functon: $F(v,\Omega)$ where $v \in \Omega \cap \tilde{\Omega}$. More complicated functions such as $\ln \partial_u f(u,v,\Omega)$ will be considered at the end.

First we must study infinitesmal perturbations. A smooth one satisfies

$$\Lambda = f(\tilde{\Omega}, v, \Omega) = \{ r e^{i\theta} \mid 0 \leq \theta < 2\pi ,\ 0 \leq r \leq 1 + \varepsilon \delta(\theta) + O(\varepsilon) \} \tag{10.1}$$

We want to compute $f(\cdot, 0, \Lambda)$ because

$$f(u, v, \tilde{\Omega}) = f[f(u, v, \Omega), 0, \Lambda] \tag{10.2}$$

Recall (5.21,23)

$$g(z) = \ln \frac{f(z, 0, \Lambda)}{z} \qquad z \in \Lambda \tag{10.3}$$

$$\operatorname{Re} g(z) = -\ln z \qquad z \in \partial\Lambda \tag{10.4}$$

The boundary condition is approximated by

$$\mathrm{Re}\, g(e^{i\theta}) = -\varepsilon\,\delta(\theta) + o(\varepsilon) \qquad (10.5)$$

This is a Dirichlet problem in $D(0,1)$ whose solution implies

$$f(z,0,\Lambda) = z\left[1 - \frac{\varepsilon}{2\pi}\int_0^{2\pi}\delta(\theta)\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta + o(\varepsilon)\right] \qquad (10.6)$$

In particular

$$F(0,\tilde{\Omega}) = F(0,\Omega)\left[1 - \frac{\varepsilon}{2\pi}\int_0^{2\pi}\delta(\theta)\,d\theta + o(\varepsilon)\right] \qquad (10.7)$$

Formula (10.7) has elegant error bounds:

__Theorem__ __10.1__: For any $0 \in \Lambda$

$$\sqrt{\frac{1}{\pi}\,\mathrm{Area}\ \mathrm{Inv}(\Lambda,0)} \le F(0,\Lambda) \le \frac{1}{2\pi}\,\mathrm{Length}\ \mathrm{Inv}(\Lambda,0) \qquad (10.8)$$

__Conjecture__:

$$F \le \frac{2}{3}\,\frac{\mathcal{L}}{2\pi} + \frac{1}{3}\sqrt{\frac{A}{\pi}} \qquad (10.9)$$

This theorem is hidden in [3]. The conjecture is ours.

Suppose we are given $\Omega \subset \tilde{\Omega}$ and required to change $\Omega(0) = \Omega$

to $\Omega(1)=\tilde{\Omega}$ continuously and monotomically

$$0\le t_1\le t_2\le 1 \Rightarrow \Omega(t_1)\subset\Omega(t_2) \tag{10.10}$$

The simplest way to do it is Loewner's method. Let the Jordan curve $P_0\subset\tilde{\Omega}\setminus\Omega$ connect $\partial\Omega$ to $\partial\tilde{\Omega}$ and define

$$P=P_0\oplus\partial\Omega \quad , \quad P_0\in S(\partial\Omega,\partial\tilde{\Omega},\tilde{\Omega}\setminus\Omega) \tag{10.11}$$

Let P be parametrized as $P(t)\ \ 0\le t\le 1$ and define

$$\Omega(t)=\tilde{\Omega}\setminus P([t,1]) \tag{10.12}$$

The change from $\Omega(t-\varepsilon)$ to $\Omega(t)$ is infinitesmal though not smooth. It turns out that (10.6) still holds with $\delta(\theta)$ a delta function

$$f(z,0,\Lambda)=z\left[1+\frac{\varepsilon\mu(t)}{2\pi}\,\frac{e^{i\sigma(t)}+z}{e^{i\sigma(t)}-z}+O(\varepsilon)\right] \tag{10.13}$$

$$\Lambda=f[\Omega(t-\varepsilon),0,\Omega(t)] \tag{10.14}$$

$$\mu(t)\ge 0 \tag{10.15}$$

The proof is simple. Clearly

$$\Lambda_0 \subset \Lambda \subset D(0,1) \tag{10.16}$$

$$\Lambda_0 = D(0,1) \setminus D[e^{i\gamma(t)}, o(1)] \tag{10.17}$$

The harmonic function from (10.3) satisfies

$$\mathrm{Re}\, g(z) = 0 \qquad z \in \partial\Lambda_0 \cap \partial D(0,1) \tag{10.18}$$

$$\mathrm{Re}\, g(z) \geq 0 \qquad z \in \partial\Lambda_0 \cap D(0,1) \tag{10.19}$$

The Dirichlet problem in Λ_0 is exactly solvable and yields (10.13). Formulas (10.13) and (10.2) for $\Omega(t-\epsilon)$, $\Omega(t)$ combine to

$$\partial_t f[u,v,\Omega(t)] = \frac{-u(t)}{2\pi} f[u,v,\Omega(t)] \frac{e^{i\gamma(t)} + f[\,''\,]}{e^{i\gamma(t)} - f[\,''\,]} \tag{10.20}$$

We have not insisted on any fixed f direction such as $\partial_1 f(0,v,\Omega) > 0$ so we can and will choose

$$\gamma(t) = 0 \tag{10.21}$$

Of course (10.21) implies an added rotation to (1.10), but we will not use them together.

The worst obsticle to the use of Theorem 1.2 to bound $F(\vartheta,\Omega)$ from below is the requirement that the comparison domain Ω_2 must contain every part of Ω no matter how far and insignificant. The following argument shows a way to throw out something.

<u>Theorem</u> <u>10.2</u>: Suppose that $\vartheta\in\Omega_1\subset\Omega_2$ where the domains are possibly multisheeted. Then

$$\frac{1-\ell^2(\partial\Omega_1\setminus\partial\Omega_2,\vartheta,\Omega_1)}{1-\ell^2(\partial\Omega_2\setminus\partial\Omega_1,\vartheta,\Omega_2)} \leq \frac{F(\vartheta,\Omega_2)}{F(\vartheta,\Omega_1)} \leq 1 \tag{10.22}$$

The inequality is sharp.

<u>Proof</u>: Let us continuously change $\Omega(0)=\Omega_1$ to $\Omega(1)=\Omega_2$ by Löewner's method. Differentiating (10.20) with respect to u results in

$$\partial_t \ln|\partial_u f(u,\vartheta,\Omega(t))| = Re\,\frac{\partial_t\partial_u f}{\partial_u f} = -\frac{\mu(t)}{2\pi}\,Re\left[\frac{2}{(1-f)^2}-1\right] \tag{10.23}$$

Inserting $u=\vartheta$ proves

$$\partial_t \ln F[\vartheta,\Omega(t)] = \frac{-\mu(t)}{2\pi} \tag{10.24}$$

and for $u\in\tilde\partial\Omega(t)$, $f[u,v,\Omega(t)] = e^{i\theta}$

$$\partial_t \ln|\partial_u f(u,v,\Omega(t))| = -\frac{\mu(t)}{2\pi}\,\frac{1}{2\sin^2\frac{\theta}{2}} = \frac{-\mu(t)}{2\pi}\,\frac{2}{|e^{i\theta}-1|^2} \qquad (10.25)$$

$$\partial_t \omega[\tilde\partial\Omega_1\cap\tilde\partial\Omega_2, v, \Omega(t)] = -\frac{\mu(t)}{2\pi}\int_{f[\tilde\partial\Omega_1\cap\tilde\partial\Omega_2, v, \Omega(t)]}\frac{d\theta}{2\sin^2\frac{\theta}{2}} \qquad (10.26)$$

Clearly

$$\min_{W\subset\partial D(0,1),\ \text{length}\,W=\omega}\int_W \frac{d\theta}{2\sin^2\frac{\theta}{2}} = \int_{\pi-\frac{\omega}{2}}^{\pi+\frac{\omega}{2}}\frac{d\theta}{2\sin^2\frac{\theta}{2}} = 2\tan\frac{\omega}{4} \qquad (10.27)$$

thus

$$\partial_t 2\ln \ell[\tilde\partial\Omega_1\cap\tilde\partial\Omega_2, v, \Omega(t)] \geq -\partial_t \ln F[v,\Omega(t)] \qquad (10.28)$$

Integration between $t=0$ and $t=1$ results in (10.22).
Equality holds iff

$$f(\Omega_1, v, \Omega_2) = D(0,1)\setminus[c,1] \qquad (10.29)$$

In practice $\ell(\tilde\partial\Omega_2\setminus\tilde\partial\Omega_1, v, \Omega_2)$ is not worth the trouble to estimate. Even when it is ignored formula (10.22) is an excellent bound when $\tilde\partial\Omega_1\setminus\tilde\partial\Omega_2$ is connected i.e. we extend Ω_1 only at one location (later we will see how to handle several locations). For example let Ω be a rectangle

$$\Omega = (-a,a) + i(-1,1) \qquad a > 0 \tag{10.30}$$

and let $\tilde{\Omega}$ be its extension into $\hat{C}$ when the edge $a + i[-1,1]$ is erased

$$\tilde{\Omega} = \hat{C} \setminus (-a + i[-1,1]) \setminus ([-a,a] - i) \setminus ([-a,a] + i) \tag{10.31}$$

Then for $a \gg 1$

$$F(0,\Omega) \sim \frac{\pi}{4}(1 + 2e^{-\pi a}) \tag{10.32}$$

$$\ell(a + i[-1,1], 0, \Omega) \sim 2e^{-\frac{\pi}{2}a} \tag{10.33}$$

and Theorem 10.2 implies

$$\frac{\pi}{4}(1 - 2e^{-\pi a}) \leq F(0,\tilde{\Omega}) \leq \frac{\pi}{4}(1 + 2e^{-\pi a}) \tag{10.34}$$

which is a very tight bound even for moderate a. The correct value is

$$F(0,\tilde{\Omega}) \sim \frac{\pi}{4}[1 + (1 - \frac{1}{e})e^{-\pi a}] \tag{10.35}$$

The extension $\tilde{\Omega}\setminus\Omega$ has a large area but little effect on $F(o,\cdot)$. Moreover most of the effect results from the part of $\tilde{\Omega}\setminus\Omega$ near the edge $a+i(-1,1)$. For instance

$$F[0,(-a,a+1)+i(-1,1)] \sim \frac{\pi}{4}[1+(1+e^{-2\pi})e^{-\pi a}] \tag{10.36}$$

The mere knowledge of $\Lambda(\upsilon,\Omega)$ determines $F(\upsilon,\Omega)$ up to a factor of 2. How much more would we learn by exsamining $\partial\Omega$'s part near υ in more detail? Combining Theorems 10.2 and 8.1 we obtain a localization theorem:

<u>Corollary</u> <u>10.3</u>: For any $\upsilon\in\Omega\setminus\infty$, $\alpha\geq 1$

$$\max\left[\frac{1}{4},\left(\frac{1-\frac{1}{\alpha}}{1+\frac{1}{\alpha}}\right)^2\right] \leq \frac{F(\upsilon,\Omega)}{F(\upsilon,D_d[\upsilon,\alpha\Lambda(\upsilon,\Omega)])} \leq 1 \tag{10.37}$$

<u>Conjecture</u>

$$\left(1+\frac{1}{\alpha}\right)^{-2} \leq \quad '' \tag{10.38}$$

Our maximal F change bound is off by at most a factor of 2.

Before proceeding any further we must make some definitions. Let A be a set of d disks each of which has a specified center even if it equals all Ω. A subset $B\subset A$ is called a μ core of A, denoted by

$$B \in T(A, \mu) \tag{10.39}$$

iff

$$\bigcup_{D \in B} D \supset \bigcup_{D(z,a,\Omega) \in A} \{z\} \tag{10.40}$$

$$\{D(z, \mu a, \Omega)\}_{D(z,a,\Omega) \in A} \text{ are disjoint} \tag{10.41}$$

where $\mu \geq 0$.

<u>Lemma</u> <u>10.4</u>: For any nonempty compact set of Ω d disks A and $0 \leq \mu \leq \frac{1}{2}$

$$T(A, \mu) \neq \{\ \} \tag{10.42}$$

<u>Proof</u>: Define a $B(A) \in T(A, \mu)$ inductively by

$$B(A) = \{D_m(A)\} \cup B[\{D(z,a) \in A \mid z \notin D_m(A)\}] \tag{10.43}$$

where $D_m(A) \in A$ maximizes the radius.

Notice that a division of (9.46-49)'s general type for moderate λ,μ is provided by any $\tilde{\mu}>0$ core of $\{D_d[\mathfrak{z}, \frac{1}{2}a(\mathfrak{z},u,v)] \mid \mathfrak{z}\in Z(u,v)\}$.

Suppose that $v\in\Omega\cap\tilde{\Omega}$. Define the perturbation boundary

$$\partial_\Delta(v,\Omega,\tilde{\Omega}) = \partial \mathrm{Com}(v,\Omega\cap\tilde{\Omega}) \setminus (\partial\Omega\cap\partial\tilde{\Omega}) \tag{10.44}$$

Define the reduced perturbation boundary by imposing a disk condition in Ω

$$\hat{\partial}_\Delta(v,\Omega,\tilde{\Omega}) = \Big\{u\in\partial_\Delta(v,\Omega,\tilde{\Omega}) \mid \exists\, p(u)\in D^c[u,\tfrac{1}{10}r_\Delta(u,\Omega,\tilde{\Omega})] \text{ and }$$

$$\big(p(u)=v \text{ or } D[p(u),\tfrac{1}{10}r_\Delta(u,\Omega,\tilde{\Omega})]\subset\Omega\big)\Big\} \tag{10.45}$$

where the perturbation radius at $u\in\partial_\Delta$ is

$$r_\Delta(u,\Omega,\tilde{\Omega}) = \max[r(u,\Omega), r(u,\tilde{\Omega})] \tag{10.46}$$

Notice that the condition inside (10.45) is automatically satisfied for any $u\in\partial_\Delta\cap\partial\tilde{\Omega}$. Our results will not be affected by imposing an extra disk condition in $\tilde{\Omega}$ or replacing (10.45) by a cone condition. For each $u\in\hat{\partial}_\Delta(v,\Omega,\tilde{\Omega})$ define the perturbation size

$$\delta_\Delta(u,v,\Omega,\tilde{\Omega}) = \begin{cases} 1-|f(u,v,\Omega)| & u\in\partial\tilde{\Omega} \qquad (10.47) \\ 1-|f(p(u),v,\Omega)| & u\in\partial\Omega \qquad (10.48) \end{cases}$$

and the perturbation disk

$$D_\Delta(u,\Omega,\tilde{\Omega}) = D_d[u, \tfrac{1}{2}r_\Delta(u,\Omega,\tilde{\Omega}),\Omega] \tag{10.49}$$

The relative F perturbation is

$$\Delta F(v,\Omega,\tilde{\Omega}) = \frac{F(v,\tilde{\Omega}) - F(v,\Omega)}{F(v,\tilde{\Omega}) + F(v,\Omega)} \tag{10.50}$$

<u>Theorem</u> <u>10.5</u>: For any $v\in\Omega\cap\tilde{\Omega}$, $\infty\notin\Omega\cup\tilde{\Omega}$, $\mu\geq 0$

$$|\Delta F(v,\Omega,\tilde{\Omega})| \leq c_1 \sum_{D_\Delta(u,\Omega,\tilde{\Omega})\in T_\Delta} \delta_\Delta^2(u,v,\Omega,\tilde{\Omega}) \leq \frac{c_2}{\mu^2} \sup_{''} \delta_\Delta('') \tag{10.51}$$

where T_Δ is any

$$T_\Delta \in T[\{D_\Delta(u,\Omega,\tilde{\Omega}) \mid u\in\partial_\Delta(v,\Omega,\tilde{\Omega})\},\mu] \tag{10.52}$$

Moreover when $\Omega\subset\tilde{\Omega}$ or $\tilde{\Omega}\subset\Omega$

$$|\Delta F(v,\Omega,\tilde{\Omega})| \geq c\mu^2 \sum_{''} \delta_\Delta^2('') \tag{10.53}$$

Proof: Clearly $\mu < 1$. We will prove Theorem 10.5 for $\Omega \subset \tilde{\Omega}$ or $\tilde{\Omega} \subset \Omega$ and obtain the general result in the following way. For any $u \in \partial\tilde{\Omega}$, $\tilde{u} \in \partial\Omega$

$$\tilde{u} \notin D[u, r(u,\Omega)] \quad , \quad u \notin D[\tilde{u}, r(\tilde{u}, \tilde{\Omega})] \tag{10.54}$$

so

$$D[u, \tfrac{1}{2} r(u,\Omega)] \cap D[\tilde{u}, \tfrac{1}{2} r(\tilde{u}, \tilde{\Omega})] = \{\,\} \tag{10.55}$$

Hence

$$T_\Delta = T_1 \cup T_2 \tag{10.56}$$

$$T_1 = T[\{D_\Delta(u, \Omega, \tilde{\Omega}) \mid u \in \partial\tilde{\Omega} \cap \partial_\Delta(v, \Omega, \tilde{\Omega})\}, \mu] =$$

$$= T[\{D_\Delta(u, \Omega, \Omega_1) \mid u \in \partial_\Delta(v, \Omega, \Omega_1)\}, \mu] \tag{10.57}$$

$$T_2 = T[\{D_\Delta(\tilde{u}, \Omega, \tilde{\Omega}) \mid \tilde{u} \in \partial\Omega \cap \partial_\Delta(v, \Omega, \tilde{\Omega})\}, \mu] =$$

$$= T[\{D_\Delta(\tilde{u}, \Omega_1, \tilde{\Omega}) \mid \tilde{u} \in \partial_\Delta(v, \Omega_1, \tilde{\Omega})\}, \mu] \tag{10.58}$$

where

$$\Omega_1 = \mathrm{Con}(v, \Omega \cap \tilde{\Omega}) \tag{10.59}$$

Theorem 10.5 applied to $\Omega \supset \Omega_1$, $\Omega_1 \subset \tilde{\Omega}$ results in

$$|\Delta F(v, \Omega, \Omega_1)| \leq C \sum_{D_d(u, \Omega, \Omega_1) \in T_1} \delta_\Delta^2(u, v, \Omega, \Omega_1) \tag{10.60}$$

$$|\Delta F(v, \Omega_1, \tilde{\Omega})| \leq C \sum_{D_d(\tilde{u}, \Omega_1, \tilde{\Omega})} \delta_\Delta^2(\tilde{u}, v, \Omega_1, \tilde{\Omega}) \tag{10.61}$$

Obviously for any $u \in \partial_\Delta(v, \Omega, \Omega_1)$, $\tilde{u} \in \partial_\Delta(v, \Omega_1, \tilde{\Omega})$

$$\delta_\Delta(u, v, \Omega, \Omega_1) = \delta_\Delta(u, v, \Omega, \tilde{\Omega}) \tag{10.62}$$

$$\delta_\Delta(\tilde{u}, v, \Omega_1, \tilde{\Omega}) \leq \delta_\Delta(\tilde{u}, v, \Omega, \tilde{\Omega}) \tag{10.63}$$

and the general Theorem follows.

Assume that $\Omega \supset \tilde{\Omega}$ and define

$$\Lambda = f(\tilde{\Omega}, v, \Omega) \tag{10.64}$$

Obviously

$$\frac{F(0,\Omega)}{F(0,\widetilde{\Omega})} = F^{-1}(0,\Lambda) \tag{10.65}$$

Let

$$T \in T[\{D_d[\zeta, 4\Lambda(\zeta), D(0,1)] \mid \zeta \in \partial\Lambda \setminus \partial D(0,1), \tfrac{1}{2}] \tag{10.66}$$

$$\Lambda(\zeta) = \Lambda[\zeta, D(0,1)] = 1 - |\zeta| \tag{10.67}$$

We will prove that

$$c_2 \sum_{D_d(\zeta) \in T} \Lambda^2(\zeta) \le 1 - F^{-1}(0,\Lambda) \le c_2 \sum_{D_d(\zeta) \in T} \Lambda^2(\zeta) \le c_3 \sup_{D_d(\zeta) \in T} \Lambda(\zeta) \tag{10.68}$$

Denote

$$T = \{D_j\}_{j \ge 1} = \{D_d[\zeta_j, 4\Lambda_j, D(0,1)]\}_{j \ge 1} \tag{10.69}$$

The upper bound on $1 - F^{-1}(0,\Lambda)$ **is obvious:**

$$F^{-1}(0,\Lambda) \ge F^{-1}[0, D(0,1) \setminus \bigcup_{j \ge 1} D_j] = \prod_{j \ge 1} \frac{F[0, D(0,1) \setminus \bigcup_{\lambda=1}^{j-1} D_\lambda]}{F[0, D(0,1) \setminus \bigcup_{\lambda=1}^{j} D_\lambda]} \ge$$

$$\ge \prod_{j \ge 1} F^{-1}[0, D(0,1) \setminus D_j] \ge \prod_{j \ge 1} (1 - c\Lambda_j^2) \ge 1 - c \sum_{j \ge 1} \Lambda_j^2 \tag{10.70}$$

and

$$2\pi \geq \sum_{j\geq 1} \mathrm{Length}[\partial \tilde{D}_j \cap \partial D(0,1)] \geq \sum_{j\geq 1} 2 r_j \tag{10.71}$$

$$\sum_{j\geq 1} r_j^2 \leq \pi \sup_{j\geq 1} r_j \tag{10.72}$$

The lower bound is a bit tricky. Replace $\{D_j\}_{j\geq 1}$ by any finite subset thereof and arrange it monotonically nondecreasing in r_j:

$$j \geq k \Rightarrow r_j \geq r_k \tag{10.73}$$

and define

$$\tilde{D}_j = D_d[\zeta_j, 2r_j, D(0,1)] \tag{10.74}$$

Then

$$F^{-1}(0,\Lambda) = \prod_{j\geq 1} \frac{F(0, \mathrm{Con}[0, D(0,1) \setminus \bigcup_{k=1}^{j-1} (\tilde{D}_k \cap \partial\Lambda)])}{F(0, \mathrm{Con}[0, D(0,1) \setminus \bigcup_{k=1}^{j} (\tilde{D}_k \cap \partial\Lambda)])} \leq$$

$$\leq \prod_{j\geq 1} \frac{F(0, P_j)}{F(0, Q_j)} = \prod_{j\geq 1} F^{-1}[0, f(P_j, 0, Q_j)] \tag{10.75}$$

where

$$Q_j = D(0, 1-2r_j) \vee e^{(-\infty,0)+i\,\mathrm{Arg}\, z_j + i(-r_j, r_j)} \tag{10.76}$$

$$P_j = \mathrm{Con}\left[0, a_j \setminus (\tilde{D}_j \cap \tilde{\partial}\Lambda)\right] \tag{10.77}$$

Clearly

$$Q_j \cap \tilde{D}_j \cap \tilde{\partial}\Lambda \in S(j, \partial a_j, a_j) \tag{10.78}$$

so by Theorem 3.3

$$F^{-1}[0, f(P_j, 0, a_j)] \leq \frac{4(1-\tilde{r}_j)}{(1+\tilde{r}_j)^2} \tag{10.79}$$

where

$$\tilde{r}_j = 1 - |f(z_j, 0, a_j)| \geq c\, r_j \tag{10.80}$$

Formulas (10.75,79,80) combine to prove the leftmost inequality of (10.68).

We will transform (10.68) to Ω. For any z_j there exists $u_j \in \tilde{\partial}\tilde{\Pi} \setminus \tilde{\partial}\Omega$ such that

$$\mathrm{inv}\, f(z_j, 0, \Omega) \in D_d(u_j, \Omega, \tilde{\Pi}) \in T_\Delta \tag{10.81}$$

Theorem 3.5 implies that

$$r_j \le \delta_\Delta(u_j, v, \Omega, \tilde{\Omega}) \tag{10.82}$$

which proves the upper bound. Moreover for any $u \in \tilde{\partial}\tilde{\Omega} \setminus \tilde{\partial}\Omega$

$$\delta_\Delta(u, v, \Omega, \tilde{\Omega}) \le c\, r[f(u, v, \Omega), D(0,1)] \tag{10.83}$$

and

$$\{E(u)\}_{D_d(u) \in T_\Delta} \quad \text{are disjoint} \tag{10.84}$$

where

$$E(u) = D[f(u), \tfrac{u}{10} r(f(u))] \tag{10.85}$$

Thus

$$\sum_{D_d(u) \in T_\Delta} \delta_\Delta^2(u) \le c_1 \sum_{j \ge 1} \sum_{\substack{D_d(u) \in T_\Delta \\ f(u) \in D_j}} \frac{\text{Area } E(u)}{\mu^2} \le \frac{c}{\mu^2} \sum_{j \ge 1} r_j^2 \tag{10.86}$$

which proves the lower bound.

Now assume that $\Omega \subset \tilde{\Omega}$. Define

$$\Lambda = f(\Omega, v, \tilde{\Omega}) \tag{10.87}$$

the construction of (10.66) is too crude for this case. We claim that either

$$r(0,\Lambda) \leq \frac{1}{2} \tag{10.88}$$

or

$$\partial\Lambda \setminus \partial D(0,1) \subset \bigcup_{D \in A} D \tag{10.89}$$

where

$$A = \{ D_{d}[\zeta, 4r(\zeta), D(0,1)] \mid \zeta \in \partial\Lambda \setminus \partial D(0,1)$$

$$\exists\, q(\zeta) \in D[\zeta, r(\zeta)] \quad D[q(\zeta), r(\zeta)] \subset \Lambda \} \tag{10.90}$$

Suppose that $r(0,\Lambda) \geq \frac{1}{2}$. For any $y \in \partial\Lambda \setminus \partial D(0,1)$ define s to be the first $0 < s < 1$ such that $\frac{y}{|y|} D^{c}(1-s, \frac{1-s}{2}) \not\subset \Lambda$. Hence

$$\frac{y}{|y|} D(1-s, \frac{1-s}{2}) \subset \Lambda \tag{10.91}$$

and there exists

$$\zeta \in \partial\Lambda \cap \frac{y}{|y|} D(\,''\,) \tag{10.92}$$

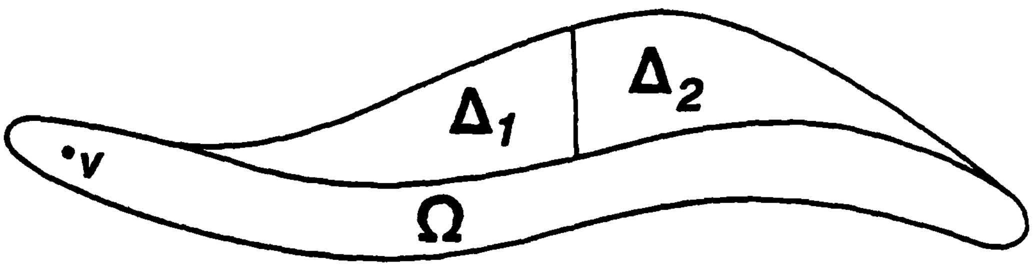
It is easy to prove that

$$y \in D[z, \sqrt{3}\, r(z)] \qquad (10.93)$$

which proves (10.90). In case (10.88) set $T = \{D_d[z, 4r(z)]\}$ where z maximizes $r(z)$. In case (10.89) set $T = T(A)$. In both cases the proof of theorem 10.5 proceeds as before. The relations between (10.45), (10.90) and the rest of the proof are the following. In order to prove (10.83) (for $\Omega \subset \tilde{\Omega}$) one needs (10.90). The area argument in (10.86) necessitates (10.45) and once (10.45) is imposed formula (10.81) relies on (10.90).

Formula (10.51) indicates that the perturbation can be broken into basic parts and that up to a constant their effects simply sum up. This should not be misinterpreted to mean that there is not much interaction between the parts. For instance, in Fig. 10.1 adding Δ_2 to Ω has far less effect than adding Δ_2 to $\Omega \cup \Delta_1$.

Theorem 10.5 estimates ΔF as accurately as can be reasonably expected except that the lower bound may be extended to one which is valid in general, though it may be in many cases because we know the amplitudes of the positive and negative contributions only up to a multiplicative constant. The detailed $\tilde{\Omega}$ structure has not been completely eliminated: $\Omega \cap \complement\tilde{\Omega}$ appears in (10.52). Still it is remarkable



Δ_1
Δ_2
$\cdot v$
Ω

that we have gotten even that close. The $\sum_{D_d(u)\in T_\Delta} \delta_\Delta^2(u)$ part corresponding to $u \in \tilde{\partial}\Omega$ can be replaced by

$$\int_{\tilde{\Omega}\cap\tilde{\partial}\mathrm{Con}(v,\Omega\cap\tilde{\Omega})} \delta(u,v,\Omega,\tilde{\Omega})\,\omega(du,\Omega) \tag{10.94}$$

which is similar to the infinitesimal formula (10.6). The rest of the sum has a similar upper but not lower bound. THe rightmost bound of $|\Delta F|$ in (10.51) follows from a fraction of the complete proof.

Define the maximal external curvature of $\tilde{\partial}\mathrm{Con}(v,\Omega\cap\tilde{\Omega})$ relative to v to be

$$K(v,\Omega,\tilde{\Omega}) = \sup_{u\in\tilde{\partial}\mathrm{Con}(v,\Omega\cap\tilde{\Omega})} [-\kappa(u,\tilde{\partial}\mathrm{Con}[v,\Omega\cap\tilde{\Omega}])\cdot d(u,v,\Omega)] \tag{10.95}$$

where $\kappa(u,\tilde{\partial}(\text{''}))$ denotes $\tilde{\partial}(\text{''})$'s curvature at u which is positive (negative) when $\mathrm{Con}(\text{''})$ is locally convex (concave). Inserting the $u\in\tilde{\partial}(\text{''})$ which minimizes $|u-v|$ proves that

$$K(v,\Omega,\tilde{\Omega}) \geq -1 \tag{10.96}$$

<u>Theorem</u> <u>10.6</u>: For any $v\in\Omega\cap\tilde{\Omega}$; $\alpha \notin \Omega\cup\tilde{\Omega}$

$$|\Delta F(\vartheta,\Omega,\tilde{\Omega})| \le c \sup_{u\in\partial\mathrm{Con}(\vartheta,\Omega\cap\tilde{\Omega})} \frac{r(\vartheta,\Omega)\, r_\Delta(u,\Omega,\tilde{\Omega})}{d^2(u,\vartheta,\Omega)} \cdot$$

$$\cdot\left[\frac{r(\vartheta,\Omega)}{d(u,\vartheta,\Omega)}+\frac{1}{K(\vartheta,\Omega,\tilde{\Omega})+2}\right]^{-1/2}\left[\frac{r_\Delta(u,\Omega,\tilde{\Omega})}{d(u,\vartheta,\Omega)}+\frac{1}{K(\vartheta,\Omega,\tilde{\Omega})+2}\right]^{-1/2} \quad (10.97)$$

Proof: It is sufficient to prove (10.97) for $\Omega\subset\tilde{\Omega}$ because then it applies to $\Omega_1=\mathrm{Con}(\vartheta,\Omega\cap\tilde{\Omega})\subset\Omega$, $\Omega_1\subset\tilde{\Omega}$ and

$$d(u,\vartheta,\Omega_1) \ge d(u,\vartheta,\Omega) \quad (10.98)$$

$$r(\vartheta,\Omega_1) \le r(\vartheta,\Omega) \quad (10.99)$$

Thus assume that $\Omega\subset\tilde{\Omega}$. Let $u\in\partial_\Delta(\vartheta,\Omega,\tilde{\Omega})$ maximize $\delta_\Delta(u,\vartheta,\Omega,\tilde{\Omega})$ and denote $u_1=p(u)$ of (10.45). Theorem 10.5 implies that

$$|\Delta F| \le c\,\delta_\Delta \le e^{-\varphi(u_1,\vartheta,\Omega)} \quad (10.100)$$

Normalize

$$d(u,\vartheta,\Omega)=1 \quad (10.101)$$

Let u_2, u_3 be the first, if any, $\Gamma(u_2, v, \Omega)$ points in $\partial D_d(u, \frac{1}{4}\frac{1}{K+2}, \Omega), \partial D_d(u, \frac{1}{2}, \Omega)$ respectively. Clearly

$$\rho(u_2, u_3) \geq \int_{\frac{2}{3}r_\Delta}^{1/2} \frac{dx}{4x} = \frac{1}{4}\ln\frac{3}{4r_\Delta} \qquad (10.102)$$

Suppose that

$$\frac{2}{3}r_\Delta < \frac{1}{4}\,\frac{1}{K+2} \qquad (10.103)$$

Then for any $z \in \tilde{\partial}\Omega \cap D_d(u, \frac{3}{4}, \Omega)$

$$-K(z, \tilde{\partial}\Omega) \leq \frac{K}{d(z, v, \Omega)} \leq 4K \qquad (10.104)$$

Thus for any $w \in D_d(u, \frac{1}{4}, \Omega)$

$$D_d(w, \tfrac{1}{2}, \Omega) \subset C \setminus D(u + \tfrac{1}{4K}\hat{n}, \tfrac{1}{4|K|}) \qquad (10.105)$$

where $\hat{n}$ is the inside normal to $\tilde{\partial}\Omega$ at u. Corollary 10.3 implies that

$$F(w, \Omega) \geq \left(\frac{1-2x}{1+2x}\right)^2 \frac{1}{2x(1+2Kx)} \geq \frac{1}{2K} - K - 4 \qquad (10.106)$$

$$x = |v - u| \qquad (10.107)$$

Hence

$$\rho(u_1,u_3)=\rho(u_1,u_2)+\rho(u_2,u_3)\geq$$

$$\geq\int_{\frac{2}{3}r_\Delta}^{\frac{1}{4}\frac{1}{K+2}}\left(\frac{1}{2x}-K-4\right)dx+\int_{\frac{1}{4}\frac{1}{K+2}}^{1/2}\frac{dx}{4x}\geq\frac{1}{4}\ln\frac{1}{K+2}+\frac{1}{2}\ln\frac{1}{r_\Delta}+c \tag{10.108}$$

Together with (10.102)

$$\rho(u_1,u_3)\geq\frac{1}{2}\ln\frac{1}{r_\Delta}+\frac{1}{4}\ln\left(r_\Delta+\frac{1}{K+2}\right)+c \tag{10.109}$$

Similarly

$$\rho(v,u_3)\geq\frac{1}{2}\ln\frac{1}{r}+\frac{1}{4}\ln\left(r+\frac{1}{K+2}\right)+c \tag{10.110}$$

Formulas (10.100,109,110) prove (10.97).

When K is not available Theorem 10.6 degenerates into

$$|\Delta F(v,\Omega,\tilde{\Omega})|\leq C\sqrt{\sigma(v,\Omega,\tilde{\Omega})} \tag{10.111}$$

where

$$\sigma(v,\Omega,\tilde{\Omega})=\sup_{u\in\partial\mathrm{Con}(v,\Omega\cap\tilde{\Omega})}\frac{r(v,\Omega)\,r_\Delta(u,\Omega,\tilde{\Omega})}{d^2(u,v,\Omega)} \tag{10.112}$$

will be called the directed distance from Ω to $\tilde{\Omega}$ relative to

ϑ . To get an idea about how σ behaves consider the four cases in Fig. 10.2.

Localization is a special kind of perturbation so Theorem 10.6 is applicable to the situation of Corollary 10.3 and implies that for any $\alpha \geq 1$

$$\left| \frac{F(\vartheta,\Omega)}{F(\vartheta, D_d[\vartheta,\alpha\Lambda(\vartheta,\Omega),\Omega])} - 1 \right| \leq \frac{c}{\sqrt{\alpha}} \tag{10.113}$$

which is inferior to the bound $\frac{c}{\alpha}$ obtained there.

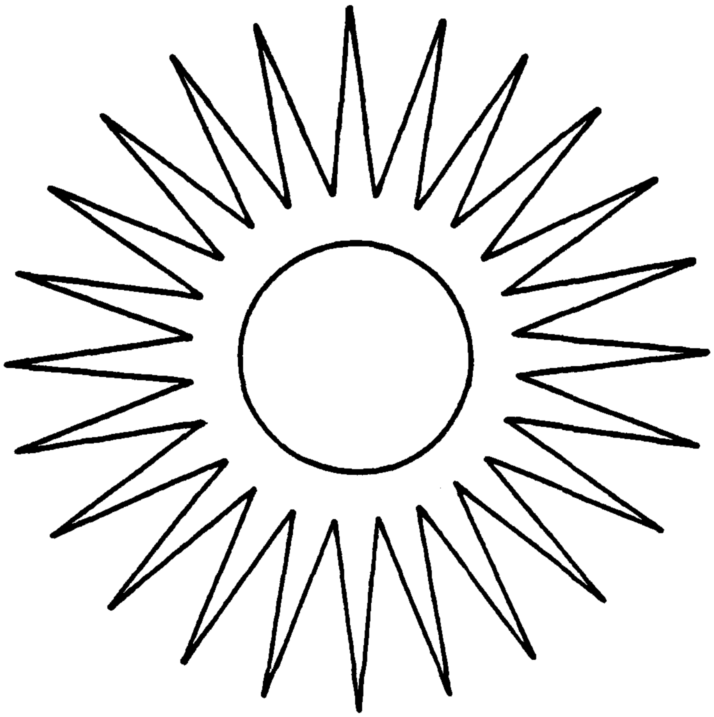
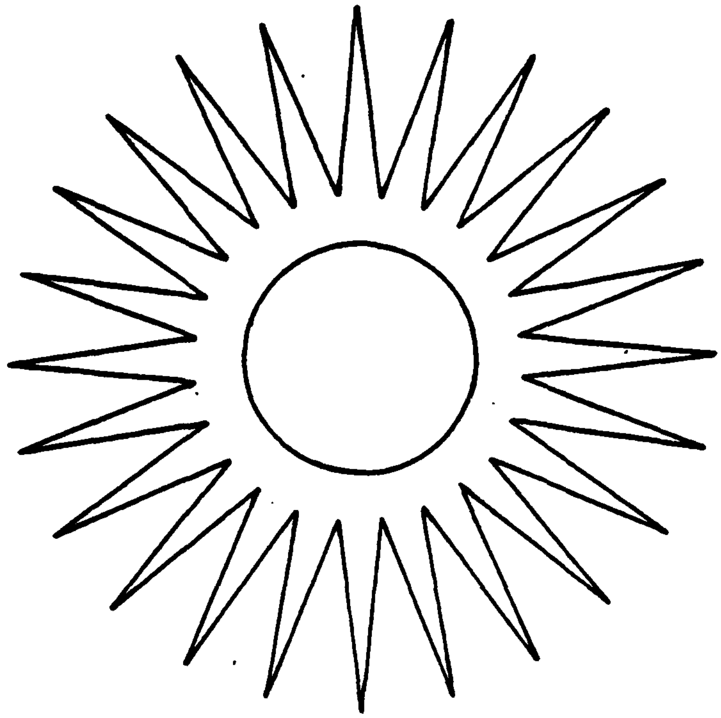
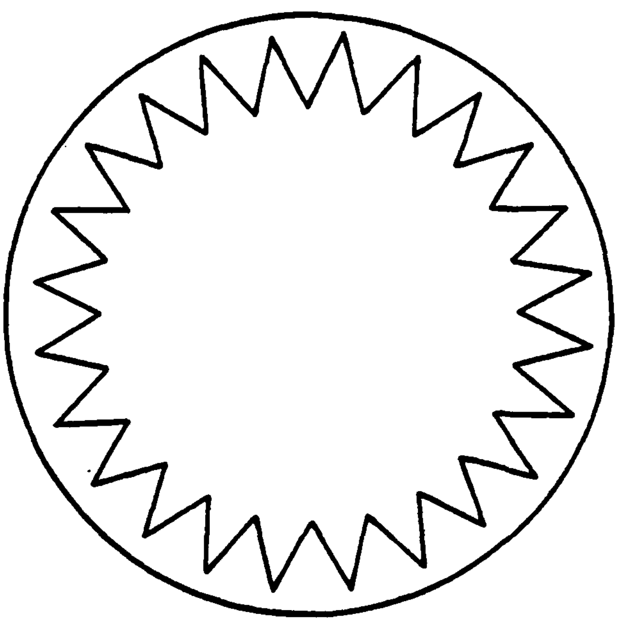
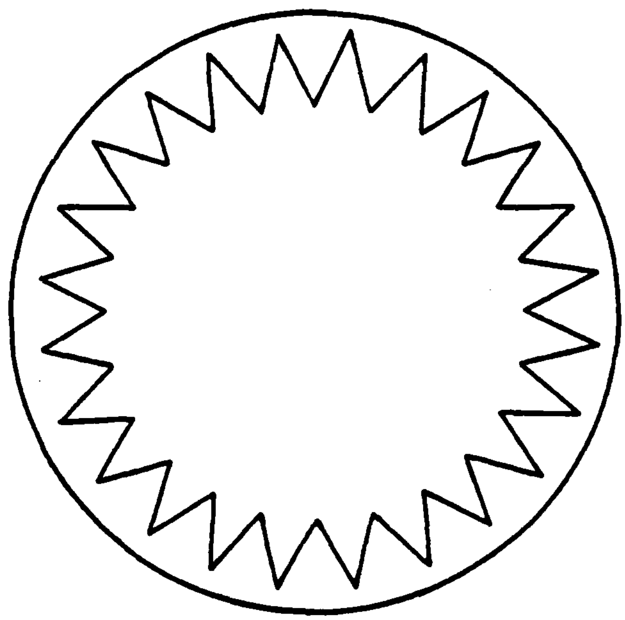
For moderate K the bound (10.97) is linear in σ . When K is large it is proportional to $\sqrt{\sigma}$. Can that happen to $|\Delta F|$? The simplest nonlinear σ dependence occures for an internal corner such as

$$\Omega = \hat{C} \setminus [-\infty, 0] \quad , \quad \vartheta = 1 \tag{10.114}$$

It is perturbed to

$$\tilde{\Omega} = \Omega \setminus \bigcup_{u \in \partial\Omega} D_d[u, \frac{\sigma}{1-\sigma} d(u,\vartheta,\Omega),\Omega] \tag{10.115}$$

The magnitude of $|\Delta F(\vartheta,\Omega,\tilde{\Omega})|$ is provided by Theorem 10.5. Let us consider only the part of $\partial\tilde{\Omega}$ near $(-1,0)$. In this case a T_Δ is easily chosen to consist of $\frac{c}{\sigma}$ disks whose

$$\delta_\Delta(u) \geq c \frac{\tilde{\sigma}}{\sqrt{\tilde{\sigma} + |u|}} \tag{10.116}$$

Thus

$$|\Delta F| \geq c \sum_{j=1}^{c/\tilde{\sigma}} \frac{\tilde{\sigma}}{j} > c\, \tilde{\sigma} \ln \tilde{\sigma} \tag{10.117}$$

A similar upper bound holds.

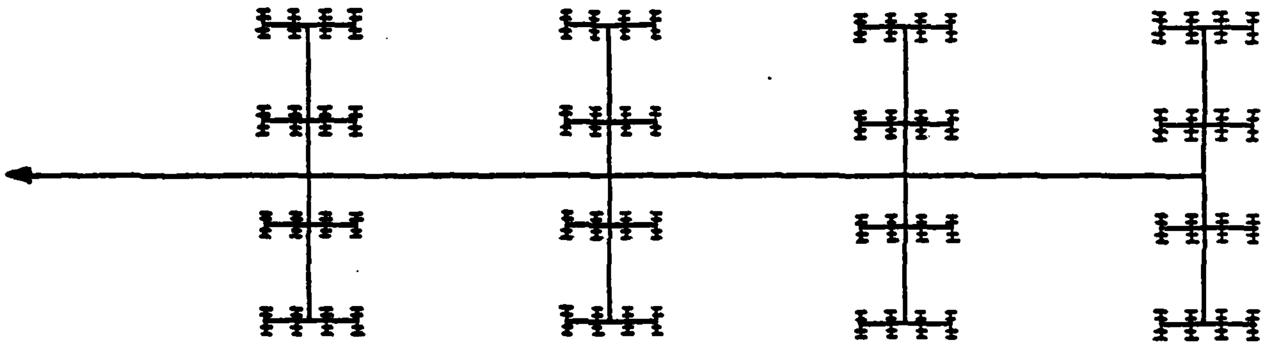
The previous example shows a nonlinear σ dependence, but the nonlinearity is extremely tame. So let us add more external curvature! The star $\Omega = \hat{C} \setminus \bigcup_{k=1}^{n} e^{ik\frac{2\pi}{n}}[1, +\infty]$ is a dismal failure. However, the fractel in Fig. 10.3 succeeds when n is large enough.

<u>Theorem</u> <u>10.7</u>: There exists a domain Ω and a $v \in \Omega$ such that for any $0 < \sigma < 1$ there exists a $\tilde{\Omega} \subset \Omega$ such that

$$\sigma(v, \Omega, \tilde{\Omega}) = \sigma \tag{10.118}$$

$$|\Delta F(v, \Omega, \tilde{\Omega})| > c\, \sigma^{\nu} \tag{10.119}$$

$$\frac{1}{2} \leq \nu < 1 \tag{10.120}$$



where ν is constant.

<u>Proof</u>: Define

$$\Omega = C \setminus (-\infty, \tfrac{1}{2}) \setminus \bigcup_{j\geq 0} P_j \tag{10.121}$$

where

$$P_0 = [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{10.122}$$

$$P_j = P_0 \cup \bigcup_{k=-n}^{n+1} \frac{1}{2n+1}(k - \tfrac{1}{2} + P_{j-1}) \tag{10.123}$$

where $n \geq 1$ will be specified. Define

$$\Omega_j = C \setminus (-\infty, -\tfrac{1}{2}) \tag{10.124}$$

Obviously

$$\sigma(1, \Omega, \Omega_{j-1}) \leq c\,(2n)^{-j} \tag{10.125}$$

Theorem 10.5 implies that

$$|\Delta F(1, \Omega_j, \Omega_{j-1})| \geq c \sum_{U \in T_j} \omega^2(U, U, \Omega_j) \tag{10.126}$$

AD-A098 067 MASSACHUSETTS INST OF TECH CAMBRIDGE DEPT OF MATHEMATICS F/G 12/1
THEORETICAL AND NUMERICAL ANALYSIS OF CONFORMAL MAPPING.(U)
JAN 81 M DUBINER AFOSR-77-3405

UNCLASSIFIED AFOSR-TR-81-0392 NL

3 of 3
AD A
098067

END
DATE FILMED
5-81
DTIC

where T_j is the set of $(P_j \setminus P_{j-1})^c$'s connected componets. We have lumped together opposite parts of $\tilde{\partial}\Omega_j$ for notational simplicity. The equivalence of $\omega(V)$ to $\delta_\Delta(u),\ u \in V$ is obvious. Theorem 3.4's physical interpretation is intuitively useful. In its terms Ω was designed to compress the charge lines.

Each $W \in T_{j-1}$ intersects $2n+2$ $V \in T_j$ which will be labled $V_1, V_2 \cdots V_{2n+2}$. We claim that

$$\omega(V_\ell, \vartheta, \Omega_j) \geq c\,\omega(W, \vartheta, \Omega_{j-1}) \cdot \omega(V_\ell, \tilde{\vartheta}, Q) \tag{10.127}$$

where

$$Q = C \setminus \overline{W \setminus \bigcup_{\ell=1}^{2n+2} V_\ell} \setminus [w + i(W - w)] \tag{10.128}$$

$$\tilde{\vartheta} = w + \hat{n}\,\cancel{\text{Length}}\,W \tag{10.129}$$

w is the center of the interval W and $\hat{n}$ is its tangent so that $\tilde{\vartheta}$ is well inside Ω_j in W's scale. The proof relies on ω's perturbation theory which will be outlined later in this section. Formula (10.127) is obtained in three steps. First for any two infinitesimal $dq_1, dq_2 \in Q$

$$\frac{\omega(dq_1,\vartheta,\Omega)}{\omega(dq_2,\vartheta,\Omega)} \Big/ \frac{\omega(dq_1,\vartheta,\Omega_j)}{\omega(dq_2,\vartheta,\Omega_j)} \leq c \tag{10.130}$$

second a localized result

$$\omega(dq,\tilde{\vartheta},\Omega_j) \geq c\,\omega(dq,\tilde{\vartheta},Q) \tag{10.131}$$

and third

$$\omega(Q,\vartheta,\Omega_j) \geq c\,\omega(W,\vartheta,\Omega_{j-1}) \tag{10.132}$$

Formula (10.127) implies

$$\sum_{k=1}^{2n+2} \omega^2(U_k,\vartheta,\Omega_j) \geq c\mu\,\omega^2(W,\vartheta,\Omega_{j-1}) \tag{10.133}$$

where

$$\mu = \sum_{k=1}^{2n+2} \omega(U_k,\tilde{\vartheta},Q) \tag{10.134}$$

is independent of j or W. It is easy to show that as in (10.117)

$$\mu \geq c\,\frac{\ln 2n}{2n} \tag{10.135}$$

By induction

$$|\Delta F| \geq C_2 \mu^{\dot{\gamma}} \geq C_2 \sigma^{\nu} \tag{10.136}$$

$$\nu = 1 - \frac{\ln \ln 2n - c}{\ln 2n} \tag{10.137}$$

<u>Conjecture</u> <u>10.8</u>: For any $v \in \Omega \cap \tilde{\Omega}$, $\infty \notin \Omega \cup \tilde{\Omega}$

$$|\Delta F(v, \Omega, \tilde{\Omega})| \leq C\, \sigma^{\nu}(v, \Omega, \tilde{\Omega}) \tag{10.138}$$

$$\tfrac{1}{2} < \nu < 1 \quad \text{constant} \tag{10.139}$$

It is time to consider other f related functions besides . There seems to be only one other independent monotone function: $\rho(u, v, \Omega)$. The conformal distance function "contains" the harmonic measure function because for any $u \in \partial\Omega$ having an inside unit normal $\hat{n}$

$$\partial_w \omega(w, v, \Omega)|_{w \downarrow u} = \lim_{\varepsilon \downarrow 0} \frac{2}{\varepsilon} e^{-2\rho(u+\varepsilon\hat{n}, v, \Omega)} \tag{10.140}$$

More generally we know that $\rho(u, v)$ and $F(v)$ determine

$|\partial_u f(u,v)|$. A third basic function is $\operatorname{Arg} f(u,v)$ where $\partial_1 f(v,v) > 0$. Together with $\rho(u,v)$ it determines $f(u,v)$ and alone it determines $\operatorname{Arg} \partial_u f(u,v)$ because

$$\operatorname{Arg} \partial_u f(u,v) = \operatorname{Arg} f(u,v) - \operatorname{Arg} f(v,u) + \pi \tag{10.141}$$

For completeness we will also consider $\partial_u^n \ln \partial_u f(u,v)$.

The entire F perturbation theory except of Theorem 10.1 is generalizable to the above mentioned functions. Of course the nonotonicity dependent upper bound of Theorem 10.2 and lower bound of Theorem 10.5 generalize only for ρ. There is no point in going over all our previous results so let us see what happens to the most detailed one, Theorem 10.5. For any $u,v \in \Omega$, $\Gamma(u,v,\Omega) \subset \tilde{\Omega}$ define the reduced perturbation boundary

$$\partial_{X\Delta}(u,v,\Omega,\tilde{\Omega}) = \bigcup_{\sigma \in \Gamma^c(u,v,\Omega)} \partial_\Delta(\sigma,\Omega,\tilde{\Omega}) \tag{10.142}$$

Notice that $\partial_{X\Delta}(u,v,\Omega,\tilde{\Omega}) \neq \partial_\Delta(v,\Omega,\tilde{\Omega})$ only when the perturbation is so large that it can hardly be called a perturbatin, a case which will be excluded. Define the perturbation size at $w \in \partial_\Delta(u,v,\Omega,\tilde{\Omega})$ to be

$$\delta_{X\Delta}(w,u,v,\Omega,\tilde{\Omega}) = \sup_{\sigma \in \Gamma^c(u,v,\Omega)} \delta_\Delta(w,\sigma,\Omega,\tilde{\Omega}) \tag{10.143}$$

Theorem 10.5': Suppose that $u, v \in \Omega$, $\Gamma^{c}(u,v,\Omega) \subset \tilde{\Omega}$ $\infty \notin \Omega \cup \tilde{\Omega}$ and

$$\sup_{w \in \gamma_{x\Delta}(u,v,\Omega,\tilde{\Omega})} \delta_{x\Delta}(w,u,v,\Omega,\tilde{\Omega}) \leq c \tag{10.144}$$

then

$$\left| \rho(u,v,\cdot)\Big|_{\Omega}^{\tilde{\Omega}} \right| \leq c \frac{\rho(u,v,\Omega)}{1+\rho(u,v,\Omega)} \sum_{D_\Delta(w,\Omega,\tilde{\Omega}) \in T_\Delta} \delta_{x\Delta}^{2}(w,u,v,\Omega,\tilde{\Omega}) \tag{10.145}$$

$$\left| [\operatorname{Arg} f(u,v,\cdot)]\Big|_{\Omega}^{\tilde{\Omega}} \right| \leq c \frac{\rho(u,v,\Omega)}{1+\rho(u,v,\Omega)} \cdot$$

$$\cdot \sum_{D_\Delta(w,\Omega,\tilde{\Omega}) \in T_\Delta} \delta_{\Delta}(w,v,\Omega,\tilde{\Omega})\, \delta_{x\Delta}(w,u,v,\Omega,\tilde{\Omega}) \tag{10.146}$$

and for any $n \geq 1$

$$\left| [\partial_u^n \ln \partial_u f(u,v,\cdot)]\Big|_{\Omega}^{\tilde{\Omega}} \right| \leq c(n) \sum_{j=1}^{n} \frac{\gamma_{n,n-j}}{\rho^{j}(v,\Omega)} \cdot$$

$$\cdot \sum_{D_\Delta(w,\Omega,\tilde{\Omega}) \in T_\Delta} \delta_{\Delta}^{j}(w,u,\Omega,\tilde{\Omega})\, \delta_{x\Delta}^{2-j}(w,u,v,\Omega,\tilde{\Omega}) \tag{10.147}$$

where T_Δ is as specified in Theorem 10.5 and

$$\gamma_{n,\ell} = \sum_{\{\nu_L\}\in I} \prod_{L=1}^{\ell} \left| \partial_u^{L} \ln \partial_u f(u,v,\Omega) \right|^{\nu_L} \leq$$

$$\leq C(n) \sum_{L=1}^{\ell} \left| \partial_u^{L} \ln \partial_u f(u,v,\Omega) \right|^{\frac{n}{L}} \tag{10.148}$$

$$I = \left\{ \{\nu_L\}_{L=1}^{\ell} \;\middle|\; \nu_L \geq 0 \;,\; \sum_{1}^{\ell} L\nu_L = n \right\} \tag{10.149}$$

Moreover when $\Omega \subset \tilde{\Omega}$ or $\tilde{\Omega} \subset \Omega$ the reverse of inequality (10.145) holds with a different constant.

Under the previous assumptions for any $j \geq 0$

$$\sum_{D_\Delta(\omega)\in T_\Delta} \delta_\Delta^{j}(\omega,u)\, \delta_{x\Delta}^{2-j}(\omega,u,v) \leq \frac{c}{\mu^2} \cdot$$

$$\cdot \min\left[\sup_{D_\Delta(\omega)\in T_\Delta} \delta_\Delta^{j-1}(\cdot\cdot)\, \delta_{x\Delta}^{2-j}(\cdot\cdot) \,,\, [1+\rho(u,v)] \sup_{D_\Delta(\omega)\in T_\Delta} \delta_\Delta^{j}(\cdot\cdot)\, \delta_{x\Delta}^{2-j}(\cdot\cdot) \right] \tag{10.150}$$

$$\frac{1}{c} \leq \frac{\delta_\Delta(\omega,u,v,\Omega)}{\rho(u,\Omega)\, \delta_{x\Delta}(\omega,u,v,\Omega)} \left| \partial_u f(u,v,\Omega) \right|^{-1} \leq c \tag{10.151}$$

where $v \in \Gamma^c(u,v,\Omega)$ maximizes $\delta_\Delta(\omega,v,\Omega,\tilde{\Omega})$.

<u>Corollary</u>:

$$\left| \left[\ln |\partial_u f(u,v,\Omega)| \right] \Big|_{\Omega}^{\tilde{\Omega}} \right| \leq c \sum_{D_\Delta(\omega)\in T_\Delta} \delta_{x\Delta}^{2}(\omega,u,v) \tag{10.152}$$

$$\left| \left[\operatorname{Arg} \partial_u f(u,v,\cdot) \right] \Big|_{\Omega}^{\tilde{\Omega}} \right| \leq c \, \frac{\Delta(u,v)}{1+\rho(u,v)} \sum_{D_\Delta(w) \in T_\Delta} \left[\delta_\Delta(w,u) + \delta_\Delta(w,v) \right] \delta_{x_\Delta}^{(1)} \qquad (10.153)$$

Condition (10.144) is the weakest reasonable formulation of the statement: $\tilde{\Omega}$ is a perturbation of Ω relative to u,v of relative size less then c . When (10.144) is violated for $u=v$ the upper bound of Theorem 10.5 is useless. Notice that when $u \in \partial\Omega$ both $\rho(u)$ and $\delta_\Delta(w,u)$ are 0 but they cancel each other as in (10.151). The $\delta_\Delta^2(w,u)$ in (10.147) implies that $\partial_u^n \ln \partial_u f(u,v)$ $n \geq 1$ is mainly affected by w's near u. In contrast $\operatorname{Arg} f(u,v)$ is mainly affected by w's near v. A more direct way to se these facts is to combine localization theory with Theorem 5.1. Each of the j indexed terms in (10.147) bounds the change in the real and imaginary parts ot the generalizations of (5.21,22).

The single most striking fact about formula (10.147) is that it disregards the smoothness of the perturbation. The situation is analogous to the following. We want to compute an integral transform in $u \in \Omega$ whose integration variable is $w \in \tilde{\partial}\Omega$. The kernel is singular at $u = w \in \tilde{\partial}\Omega$ and we know only the order of magnitude $\rho_\Delta(w)$ of the transformed function and its sign. Formula (10.147) is the best possible under these circumstances. The reverse of inequality (10.145) holds with a different constant when $\Omega \subset \tilde{\Omega}$ or $\tilde{\Omega} \subset \Omega$ because then the integrand is of a fixed sign. a great amount of cancelation occures when u is near $\tilde{\partial}\Omega$, most of the contribution to the sum comes from u's immediate neighborhood and the perturbation is smooth. In that situation (10.147) is of little use.

11. <u>Numerical Conformal Mapping</u>.

Suppose that we are given a simply connected domain Ω. In the easiest to simulate incompressible fluid problemes is periodic so it can be scaled and rotated to satisfy

$$\Omega + 2\pi = \Omega \tag{11.1}$$

$$\exists c \quad c + i\,D(\infty) \subset \Omega \tag{11.2}$$

which will be assumed from now on. We want to conformally map Ω onto the half plane $D(\infty)$. Let us parametrize $\tilde{\partial}\Omega$ by $-\infty \leq x < \infty$ as

$$\tilde{\partial}\Omega = g([-\infty, \infty)] \tag{11.3}$$

$$g(x + 2\pi) = g(x) + 2\pi \tag{11.4}$$

Several parametrizations desined to resolve $\tilde{\partial}\Omega$ will be given later. Define

$$\theta(x) = f[g(x), \infty, \Omega] \tag{11.5}$$

The functions $g-\theta, \ln \partial_\theta g$ are analytic in θ and bounded at $-\infty$ (compare with (5.21,22)) so

$$(I-iH)(g-\theta) = c \tag{11.6}$$

$$(I-iH) \ln \partial_\theta g = 0 \tag{11.7}$$

where I is the identity transform and H is the Hilbert transform

$$[H\psi](x) = \frac{-1}{2\pi}\int_0^{2\pi} \frac{\psi(x')-\psi(x)}{\tan\frac{\theta(x')-\theta(x)}{2}}\, d\theta(x') \tag{11.8}$$

Manikoff and Zemack [10] took the imaginary part of (11.6)

$$\operatorname{Im} g = H(\operatorname{Re} g - \theta) + c \tag{11.9}$$

and rewrote (11.8) as

$$H\psi(x) = H_x\psi(x) + \frac{1}{2\pi}\int_0^{2\pi} \ln \frac{\sin\frac{\theta(x')-\theta(x)}{2}}{\sin\frac{x'-x}{2}}\, d\psi(x') \tag{11.10}$$

where H_x is the hilbert transform in x. Actually they have used different notation and have parametrized $x = \operatorname{Re} g$. Anyway when g is specified (11.9,10) is a nonlinear integral equation in θ. Notice that it is well behaved even when θ

is an extreme contraction of X. Manikoff and Zemack numerically approximated (11.9,10) and solved it by Newton iterations where an N point approximation requires $\mathcal{O}(N^2)$ memory locations and $\mathcal{O}(N^3)$ operations per iteration.

The MZ method has another fault, minor incomparison to the $\mathcal{O}(N^2)$ storage. The numerical separation of H into H_x and $H-H_x$ may strongly increase the influence of $\psi(x')$ on $\partial_x H\psi(x)$. We consider the ∂_x derivative of $H(\mathrm{Re}\, g-\theta)=\mathrm{Im}\, g$ because the shape of $\partial\Omega$ near $g(x)$ does not depend on an additive constant. Moreover for any harmonic function ψ

$$\partial_n\psi=\frac{1}{|\partial_x g|}\,\partial_x H\psi \tag{11.11}$$

where $\partial_n\psi$ is ψ's derivative in the inner normal direction. Clearly

$$\partial_x H\psi(x)=\frac{1}{4\pi}\int_0^{2\pi}\frac{\psi(x')-\psi(x)}{\sin^2\frac{\theta(x')-\theta(x)}{2}}\,\partial_{x'}\theta(x')\,\partial_x\theta(x)\,dx' \tag{11.12}$$

so $\psi(x')$'s influence on $\partial_x H\psi(x)$ is proportional to

$$\frac{\partial_{x'}\theta(x')\,\partial_x\theta(x)}{|\theta(x')-\theta(x)|^2} \tag{11.13}$$

which can be much smaller then the $\frac{1}{|x'-x|^2}$ influence of $\psi(x')$ on $\partial_x H_x\psi(x)$. In other words the MZ method is not conformally local. That holds with venagence for vortex methods.

Like the previous approach, our scheme is a perturbation method, but it is explicit. Soppose that we have the functions g, θ and want to perturbe Ω onto $\tilde{\Omega}$. For time dependent domains $\Omega(t)$ the perturbation is infinitesmal and one may write $g \to g + h\, dt$ as

$$D_t g = h \qquad (11.14)$$

where h is the velocity and D_t is the substatial derivative. We willadjust $\partial\Omega$'s scalling by

$$\partial_t = D_t + n\,\partial_x \qquad (11.15)$$

When iterating towards fixed $\tilde{\Omega}, \tilde{g}$ one simply defines

$$h = \frac{\tilde{g} - g}{\Delta t} \qquad (11.16)$$

where Δt is arbitrary, and performs Euler time stepping on (11.14) and and other equations. The well known formula (10.6) can be written as

$$D_t \theta = Re\,(I - iH)\,\frac{h}{\partial_\theta g} \qquad (11.17)$$

It can not be used directly because where θ croweds $D_t \theta$ can be larger than $\partial_x\theta$ by many orders of magnitude. Our first numerical observation has been that when (11.17) is differetiated by θ it gives

$$D_t \ln \partial_x\theta = \partial_\theta D_t\theta + \partial_x \Omega =$$

$$= Re(I-iH)\left[\frac{1}{\partial_x g}\left(\partial_x h + h\,\partial_x \ln \frac{\partial_x\theta}{\partial_x g}\right)\right] \tag{11.18}$$

which is well behaved even for complicated domains. We were led to (11.18) by the theoretical observation that $\ln \partial_u f(u)$ and not $f(u)$ is the correct function to consider.

Notice that as far as Ω's shape is concerned h's tangential component $Re\left(h \frac{|\partial_x g|}{\partial_x g}\right)$ is arbitrary. However it is numerically advantagious for it to be derived from either the physical velocity or (11.16) so that singularities move at velocity h. That advantage is realized by replacing $h(x')$ inside the square brackets of (11.18) with $h(x') - h(x)$. Notice the improvement near a corneror for a small scale structure on which most of h is translation. The modified (11.18) is locally translation invarient.

It is time to list our conformally local numerical implementation. We will choose $\Omega(x,t)$ from (11.15) to be a sum of delta functions in the time t so that within each time step $\Omega=0$ and at its end the scalling is reset abruptly without any time stepping errors. All the functions g, θ etc.

will be computed at the N points

$$x_\ell = \ell \Delta x = \ell \frac{2\pi}{N} \qquad 0 \le \ell \le N-1 \tag{11.19}$$

and for an arbitrary function of x $\psi(x)$ we will denote

$$\psi_\ell = \psi(x_\ell) \tag{11.20}$$

We time march the $2N+1$ variables

$$\{\mathrm{Re}\, g_\ell\}_{\ell=0}^{N-1}, \ \mathrm{Av}\ \ln g, \ \{\ln \Delta_x \theta_\ell\}_{\ell=0}^{N-1} \tag{11.21}$$

where

$$\mathrm{Av}\ \psi = \frac{1}{2\pi} \int_0^{2\pi} \psi(x)\, d\theta(x) \approx \frac{1}{N} \sum_\ell \psi_\ell \Delta_x \theta_\ell \tag{11.22}$$

$$\Delta_x \psi_\ell = \psi_{\ell+1} - \psi_\ell \tag{11.23}$$

by

$$D_t\, \mathrm{Re}\, g_\ell = \mathrm{Re}\, h_\ell \tag{11.24}$$

$$D_t\, \mathrm{Av}\ \ln g_\ell = \mathrm{Av}\ \ln\left(h_\ell \frac{\overline{\partial_x g_\ell}}{\partial_x g_\ell} \right) \tag{11.25}$$

$$D_t \ln \Delta_x \theta_\ell = \frac{1}{\Delta_x \theta_\ell} \Delta_x H \left[\partial_x \theta \cdot \mathrm{Im} \frac{h - \frac{h_\ell + h_{\ell+1}}{2}}{\partial_x g} \right]_\ell +$$

$$+ \mathrm{Re} \left[\Delta_x h_\ell \cdot \frac{1}{2} \left(\frac{1}{\partial_x g_\ell} + \frac{1}{\partial_x g_{\ell+1}} \right) \right] + c \tag{11.26}$$

where the constant c is determined by

$$\sum_\ell \Delta_x \theta_\ell \big|_{t+\Delta t} = 2\pi \tag{11.27}$$

The imaginary part of g is constructed by

$$\mathrm{Im}\, g_\ell = H(\mathrm{Re}\, g - \theta)_\ell + Av(\mathrm{Im}\, g)_\ell \tag{11.28}$$

The numerical $\partial_x \psi$, $H\psi$ of nice functions ψ, and $\partial_x \theta$ $\Delta_x H(\psi \partial_x \theta)$ are

$$\partial_x \theta_\ell = \left[\frac{1}{2} \sum_{\ell' \neq \ell} (-1)^{\ell' - \ell + 1} \frac{\sin(x_{\ell'} - x_\ell)}{\tan \frac{\theta_{\ell'} - \theta_\ell}{2}} \right]^{-1} \tag{11.29}$$

$$\partial_x \psi_\ell = \frac{1}{2} \sum_{\ell' \neq \ell} (-1)^{\ell' - \ell + 1} \frac{\psi_{\ell'} - \psi_\ell}{\tan \frac{\theta_{\ell'} - \theta_\ell}{2}} \cdot$$

$$\cdot \frac{\sin \frac{x_{\ell'} - x_\ell}{2}}{\sin \frac{\theta_{\ell'} - \theta_\ell}{2}} \partial_x \theta_{\ell'} \partial_x \theta_\ell \tag{11.30}$$

$$H\psi_l = -\frac{1}{N}\sum_{l'\neq l}\frac{\psi_{l'}-\psi_l}{\tan\frac{\theta_{l'}-\theta_l}{2}}\,\partial_x\theta_{l'}\,\cdot$$

$$\cdot\left[1+(-1)^{l'-l+1}\frac{\sin\frac{x_{l'}-x_l}{2}}{\sin\frac{\theta_{l'}-\theta_l}{2}}\,\partial_x\theta_l\right] \qquad (11.31)$$

$$\frac{\Delta_x H(\psi\partial_x\theta)_l}{\sin\frac{\Delta_x\theta_l}{2}} =$$

$$= -\frac{1}{N}\sum_{l'\neq l,\,l+1}\frac{\psi_{l'}\partial_x\theta_{l'}-\frac{\psi_l\partial_x\theta_l+\psi_{l+1}\partial_x\theta_{l+1}}{2}-\frac{\Delta_x(\psi\partial_x\theta)_l}{2\sin\frac{\Delta_x\theta_l}{2}}\sin\left(\theta-\frac{\theta_l+\theta_{l+1}}{2}\right)}{\sin\frac{\theta_{l'}-\theta_l}{2}\,\sin\frac{\theta_{l'}-\theta_{l+1}}{2}}\,\cdot$$

$$\cdot\left[\partial_x\theta_{l'}+(-1)^{l'-l}\frac{1}{2}\,\frac{\partial_x\theta_l+\partial_x\theta_{l+1}}{\sin\frac{\Delta_x}{2}}\sin\left(x_{l'}-x_l-\frac{\pi}{N}\right)+\Delta_x\partial_x\theta_l\right] \qquad (11.32)$$

Formulas (11.29-32) are conformally local and are accurate to an infinite order in $\frac{1}{N}$. See the Appendix about (11.29).

At the end of each time step a rescalling is done. Suppose that

$$x_l \to \tilde{x}_l \qquad (11.33)$$

Then

$$\psi(\tilde{\chi}_\ell) = \frac{\sum_{\ell'} (-1)^{\ell'-\ell+1} \dfrac{\psi_{\ell'}}{\tan \frac{\chi_{\ell'}-\tilde{\chi}_\ell}{2}} \dfrac{\sin^2 \frac{\chi_{\ell'}-\chi_\ell}{2}}{\sin^2 \frac{\theta_{\ell'}-\theta_\ell}{2}} \partial_x \theta_{\ell'}}{\sum_{\ell'} (-1)^{\ell'-\ell+1} \dfrac{1}{\tan \frac{\chi_{\ell'}-\tilde{\chi}_\ell}{2}} \dfrac{\sin^2 \frac{\chi_{\ell'}-\chi_\ell}{2}}{\sin^2 \frac{\theta_{\ell'}-\theta_\ell}{2}} \partial_x \theta_{\ell'}} \tag{11.34}$$

The points $\tilde{\chi}_\ell$ are specifiedby some scaling formula. For instance

$$\Delta_x \tilde{\chi} = c\, \nu *$$

$$* \left[|\Delta_x g|^{1-\mu} \left(0.2 |\Delta_x g|^2 + (\eta \mathrm{Re}^2 + \mathrm{Im}^2) \Delta_x \ln \frac{\partial_x \theta}{\partial_x g} \right)^{\mu/2} \right] \tag{11.35}$$

where

$$0 \le \mu, \eta \le 1 \tag{11.36}$$

are constants and $\nu *$ is a smoothing operation. The constant is determined by

$$\sum_\ell \Delta_x \tilde{\chi}_\ell = 2\pi \tag{11.37}$$

Notice that (11.35) is constructed so that $\tilde{\chi}$ is not much dependent on χ. Particular choices are $\mu = 0$ where χ approximates a constant times $\partial\Omega$'s arc length and $\mu = 1, \eta = 0$ where $\{g_\ell\}$'s density is approxmetely proportional to $\partial\Omega$'s curvature when it is large.

For incompressible irotational flow problems the velocity is determined by

$$h = \frac{\partial_x (I - iH)\phi}{\overline{\partial_x g}} \tag{11.38}$$

where the potential function ϕ evolves according to

$$D_t \phi = \frac{1}{2}|h|^2 - \mathrm{Im}\, g \tag{11.39}$$

Our numerical method (11.19-39) has not been implemented yet. An early version based on (11.7) and (11.18) with the standard spectral ∂_x has been programed. It did very well on time dependent domains where h was given aperiori, including a saw teeth domain. However when applied to the Rayleigh-Taylor instability it developed an explosive numerical instability when the spike's tip resolution became poor. The same behaviour has been shown by the Manikoff Zemack method and the vortex method but the vortex method blows up at a later time than the MZ method, and (11.18) performs similarly to MZ with much fewer memory locations and operations. We hope that (11.19-39) will do better. At worst it is flexiable enough to be modifiable into something better. Conformal locality will be certainly usefull in truely complicated problems, when we do not want a poorly resolved part of $\partial\Omega$ to contaminate others.

We still have two points to mention. One is the neccesity for specifying the center v of the conformal map. The conformal mapping is used only to compute $\partial_n \psi$ by (11.11), and it is center independent. The choice $v=\infty$ in (11.5) is highly natural because of (11.2), but can it be avoided in general? The Schwazian derivative

$$(2\partial_x - I)\, \partial_x \ln \partial_x \theta + (\partial_x \theta)^2 \tag{11.40}$$

is center independent, but we do not see any good comming out of it.

What about muliconnected domains? The standard cannonical domains have corners but the following choice of a cannonical n connected domain

$$D(0,1) \setminus \bigcup_{j=1}^{n-1} D^c(\alpha_j + i\beta_j, a_j) \tag{11.41}$$

$$\alpha_1 = \beta_1 = \alpha_2 = 0 \quad , \quad a_j \geq 0 \tag{11.42}$$

is smooth and treats all $\partial\Omega$'s components on similar footing. However the Poisson kernel depends on 2 real variables for doubly connected domains, 4 for $n=3$ and $3n-5$ for $n \geq 3$. One way of computing the kernel for $n > 2$ is to forget (11.41) and replace Ω by its multisheeted cover Ω_*. Numerically only a finite number of sheets can be considered. In order to reach accuracy ε one has to take $c(\Omega) \ln \frac{1}{\varepsilon}$ sheets. These sheets contain $N c(\Omega) \ln \frac{1}{\varepsilon}$ points for $n=2$ and an order of

$$\varepsilon^{-N c(\Omega) \ln (n-1)} \tag{11.43}$$

points for $n \geq 3$. This is unacceptable for $n \geq 3$. One can of coarse make $n-2$ cuts in the domain and solve a $M \times M$ system of equations where M is the number of points on the cuts.

Appendix: Spectral Multiscaled Integration.

Formula (11.29) has been desined to be spectral (i.e. of an infinite order of accuracy in $\frac{1}{N}$) and conformally local. In detail, suppose that we purturbe $\Delta_x\theta_l$ by a relative amount $\varepsilon \ll 1$

$$\ln \Delta_x\theta_l \rightarrow \ln \Delta_x\theta_l + \varepsilon \tag{A.1}$$

$$\ln \Delta_x\theta_{l'} \rightarrow \ln \Delta_x\theta_{l'} + \varepsilon a \qquad l' \neq l \tag{A.2}$$

$$a = \frac{1}{\varepsilon} \ln\left(1 - \frac{e^{\varepsilon}-1}{2\pi - \Delta_x\theta_l} \Delta_x\theta_l\right) = \mathcal{O}(\Delta_x\theta_l) \tag{A.3}$$

Then the effect transmited to $\partial_x\theta_{\lambda}$ by (11.29) is

$$\ln \partial_x\theta_{\lambda} \rightarrow \ln \partial_x\theta_{\lambda} + \mathcal{O}\left(\varepsilon \frac{\Delta_x\theta_{\lambda}\,\Delta_x\theta_l}{|\theta_{\lambda} - \theta_l|^2}\right) \tag{A.4}$$

which agrees with (11.13) up to the hidden unavoidable factor of N.

Formula (11.29) takes $\mathcal{O}(N^2)$ operations to compute versus $\mathcal{O}(N \ln N)$ operations for the usual spectral $\hat{\partial}_x$, but the lateris useless for multiscaled θ. Now let us replace (A.4) by the weakest acceptable requirment

$$\ln \tilde{\partial}_x \theta_k \to \ln \tilde{\partial}_x \theta_k + O(\varepsilon) \tag{A.5}$$

which is not conformally local, but still works. We now pose a question: What is the fastest scheme to acheive (A.5)? The first natural try is

$$\Delta_x \tilde{\partial}_x \theta = \Delta_x \theta \cdot \hat{\partial}_x \ln \Delta_x \theta \tag{A.6}$$

$$\sum_k \Delta_x \tilde{\partial}_x \theta_k = 0 \tag{A.7}$$

but it violates (A.5) for multiscaled θ .

We will now construct a spectral multiscaled method of integrating $\partial_x \theta$ to $\tilde{\theta}$. It can be iterated to compute differetation, but it is as valueable in its own right. Define the functions

$$\phi(j,x) = \tilde{\theta}(x + 2^{-j}\tfrac{2\pi}{N}) - \tilde{\theta}(x) \qquad j \geq 0 \tag{A.8}$$

Then clearly

$$\Delta_x \tilde{\theta}_k = \phi(x)_k \tag{A.9}$$

$$\phi(j-1)_l = \frac{1}{2}\left[\phi(j)_l + e^{[\ln\phi](j)_{l+\frac{1}{2}}}\right] \qquad \text{(A.10)}$$

and $\ln\phi$ is a nice function so $\ln\phi(j)_{l+\frac{1}{2}}$ can be computed from $\{\ln\phi(j)_{l'}\}$ in the usual spectral way. Thus it takes $O(JN\ln N)$ operations do derive $\Delta_x\theta$ from $\phi(J)$. Obviously

$$\phi(J)_l = \int_{l\frac{2\pi}{N}}^{(l+2^{-J})\frac{2\pi}{N}} e^{\ln\partial_x\theta(x)}\,dx \qquad \text{(A.11)}$$

can be computed by a K points Gaussian integration if

$$J \geq J_0 - \log_2 N + \frac{k}{O(1)} \qquad \text{(A.12)}$$

$$J_0 = \log_2\left(\sup\frac{1}{\partial_x\theta}\right) \qquad \text{(A.13)}$$

We assume that

$$J_0 \ll N \qquad \text{(A.14)}$$

The total number of operations is

$$n = O\left[(J+K)\,N\ln N\right] \qquad \text{(A.15)}$$

and assuming one dominant wave number the accuracy ε has the controlling factor

$$\ln \frac{1}{\varepsilon} = O[(J-J_0+\log_2 N)K] \tag{A.16}$$

Hence optimaly

$$n = \begin{cases} \left(J_0-\log_2 N+2\sqrt{\ln \frac{1}{\varepsilon}}\right) N \ln N & J_0-\log_2 N+\sqrt{\ln \frac{1}{\varepsilon}} > 0 \quad \text{(A.17)} \\ \dfrac{\ln \frac{1}{\varepsilon}}{\log_2 N - J_0} & \quad '' \quad < 0 \quad \text{(A.18)} \end{cases}$$

We do not have any non iterative differentiation method of type (A.10) or a non iterative integration method of type (11.29). Moreover (A.9-11) is unlikly to be the last word in efficiency.

Index of Notation

The following notation is used throughout this thesis. However we could not resist using some of the letters (such as a, α, δ, z) for other purposes which are specified on location. Sometimes functions are abbreviated by dropping some of their last arguments, for example $\delta(w, u, v, \Omega)$ to $\delta(w)$ or even δ. We may olso drop a middle argument by replacing it with a ". In all such cases the missing argument's value is the one most recently listed inside the same function with the same specified arguments. The letter c with or without indices denotes a constant. No connection is assumed between two c 's in the same formula, not to mention adjacent lines.

$\mathbb{C}$	Complex plane
$\hat{\mathbb{C}}$	Comlex plane with infinity
$\{\ \}$	Empty set
Q^c	Closure of the set in
Q^o	Open of the set
(a, b)	Open interval between a and b where $a, b \in \hat{\mathbb{C}}$

$P \setminus Q$		$\{u \mid u \in P, u \notin Q\}$
$P - Q$		$\{u, v \mid u \in P, v \in Q\}$
$Con(P, Q)$		Connected components of the set Q intersecting the set P
$Con(u, Q)$		$Con(\{u\}, Q)$ where u is a point
$\partial\Omega$		Boundary of Ω in $\hat{C}$
$\tilde{\partial}\Omega$	(2.1)	Conformal boundary
Ω^b		$\Omega \cup \tilde{\partial}\Omega$
$\partial_\Delta(v, \Omega, \tilde{\Omega})$	(10.44)	Perturbation boundary
$\eth_\Delta(v, \Omega, \tilde{\Omega})$	(10.45)	Reduced perturbation boundary
$\eth_{\times\Delta}(u, v, \Omega, \tilde{\Omega})$	(10.142)	
∂_j		Partial Derivative with respect to the j'th argument
∂_u	(0.14)	where u is complex
Ω_*	(3.19)	Cover domain
$Tcov(u, \Omega)$	(3.20)	Cover map
$r(v, \Omega)$	(3.9)	Minimal radius of Ω at v
$r_\Delta(u, \Omega, \tilde{\Omega})$	(10.46)	Perturbation radius
$\alpha(u, v, P)$	(5.29)	Angle change of a curve P from u to v where $u, v \in P$
$\kappa(u, P)$		Curvature of the curve P at u
$K(v, \Omega, \tilde{\Omega})$	(10.95)	
$\sigma(v, \Omega, \tilde{\Omega})$	(10.112)	
ε slender	(0.4,5)	

ε conjugation	(0.19,20)	
$d(u,v,\Omega)$	(7.1)	Internal Euclidian distance
$d_0(u,v,\Omega)$	(7.84)	
$D(\infty)$		The half plane
$D(v,a)$		Disk
$D_d(v,a,\Omega)$	(7.10)	d disk
$D_0(v,a,\Omega)$	(7.86)	d_0 disk
$D_\Delta(u,\Omega,\tilde{\Omega})$	(10.49)	Perturbation disk
$b(w,u,v,\Omega)$	(7.52)	bottleneck's width
$a(w,u,v,\Omega)$	(9.2)	
$\hat{n}(w,u,v,\Omega)$	(7.73,74)	
$Z(u,v,\Omega)$	(7.1)	Line of least Euclidian distance connecting u and v in Ω
$\Gamma(u,v,\Omega)$	(1.6)	Geodesic
$C\Gamma(u,v,\Omega)$		Continuation of beyond and
$T\Gamma(u,v,\Omega)$		Total geodesic $= \Gamma^c_v C\Gamma$
$f(u,v,\Omega)$		Conformal mapping function
$F(u,\Omega)$	(1.5)	Conformal metric scalar
$\rho(u,v,\Omega)$	(1.6)	Conformal metric
$Inv(v,\Omega)$	(3.5)	
$V(\Lambda)$	(3.23,24)	Capacity
$\omega(W,v,\Omega)$	(1.25)	Harmonic measure
$\lambda(W,v,\Omega)$	(6.1)	Modified harmonic measure
$S(u,v,\Omega)$	(1.7)	
$S(U,V,\Omega)$	(6.2)	

$S^*(U,V,\Omega)$	(6.11)	
S^*	(6.6)	
$\lambda(S)$	(6.2)	Extremal length
$\lambda(U,V,\Omega)$	(6.12)	
$\delta(u,a,v,\Omega)$	(8.85)	
$\delta_\chi(w,a,u,v,\Omega)$	(0.66)	
$\delta_\Delta(w,v,\Omega,\tilde{\Omega})$	(10.47,48)	Perturbation size
$\delta_{\chi\Delta}(w,u,v,\Omega,\tilde{\Omega})$	(10.143)	Generelized perturbation size
$\Delta F(v,\Omega,\tilde{\Omega})$	(10.50)	Relative F perturbatin size
$T(A,\mu)$	(10.39-41)	μ core of A

Bibliography

1. Lars V. Ahlfors, Complex Analysis
   McGraw-Hill (1966)

2. Einar Hille, Analytic Function Theory
   Ginn and Co. (1959)

3. Albert Pfluger, Lectures on Conformal Mapping
   Indiana University (1969)

4. Z. Nehari, Conformal Mapping
   McGraw-Hill (1952)

5. James A. Jenkins, Univalent Functions and Conformal Mapping
   Springer-Verlag (1965)

6. Lars V. Ahlfors, Conformal Invariants
   McGraw-Hill (1973)

7. Lars V. Ahlfors, Lectures on Quasiconformal Mappings
   Van Nostrand (1966)

8. L. L. Helms, Introduction to Potential Theory
   Wiley-Interscience (1969)

9. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis
Interscience (1958)

10. R. Menikoff and C. Zemach, Methods for Numerical Conformal Mappings
J. Comp. Phys. (1980)

11. D. I. Meiron, S.A. Orszag and M. Israeli, Applications of Numerical Conformal Mapping
J. Comp. Phys. To be published (1981)

12. G. R. Baker, D. I. Meiron and S. A. Orszag, Vortex Simulation of the Rayleigh-Taylor Instability
Phys. of Fluids (1980)

Biographical Note

Name: Moshe Dubiner.

Institutes Attended: Tel Aviv University (Undergraduate Student in Mathematics) 1972-75;
Massachusettes Institute of Technology (Graduate Student in Mathematics) 1975-80.

Degrees: B.Sc. in Mathematics, Tel Aviv University (1974).

DATE

FILMED

-8